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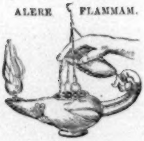
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PROCEEDINGS

OF

THE PHYSICAL SOCIETY

OF LONDON.

I. *The Sine Condition in Relation to the Coma of Optical Systems.* *By* S. D. CHALMERS, *M.A.*[*]

THE condition for the correction of coma in a centred optical system is the well-known Sine Condition. This has been proved by Clausius, Helmholtz, Hockin, and others, and the importance of this condition in the design of optical systems has been pointed out by Abbe, Steinheil, Conrady, and others; but so far as I am aware no discussion of the effects of failure to satisfy this condition has been published.

Steinheil appears to have arrived at a correct estimate of the errors in telescope systems, on the basis of a number of trigonometrical calculations.

I propose to obtain the relation between the Coma of a system and the errors in the Sine Condition.

If O and A be the angles which the initial and final rays make with the axis, and q the theoretical magnification of the system, the Sine Condition may be stated—

$$\frac{1}{q}\frac{\mu_0 \sin \mathrm{O}}{\mu \sin \mathrm{A}} - 1 = 0 \text{ for all values of the angle O,}$$

μ_0 and μ being the refractive indices of the first and last media.

We shall consider centred optical systems only and take the axis as the axis of z, and the object may be assumed in the plane $y=0$.

---

[*] Read June 25 1909.

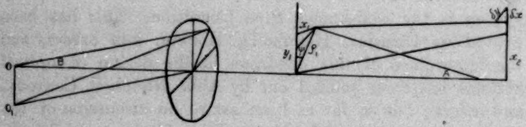
The length of the theoretical image will be denoted by x and the errors, in the directions x and y, for the actual ray, will be denoted by δx and δy, and $\frac{\delta x}{x}$ and $\frac{\delta y}{x}$ will specify the Coma for the ray considered. From approximate theories we can see that these terms depend on the particular ray considered and may be represented as follows :—

$$\frac{\delta x}{x} = i_1(2\cos^2\psi + 1)\sin^2 A + i_2(4\cos^2\psi + 1)\sin^4 A$$
$$+ \text{ terms of higher order in } \sin A,$$

$$\frac{\delta y}{y} = i_1(2\sin\psi\cos\psi)\sin^2 A + i_2(4\sin\psi\cos\psi)\sin^4 A$$
$$+ \text{ terms of higher order},$$

where ψ is the angle between the $x\ z$ plane and the radius to the point of the stop through which the ray passes (fig. 1).

Fig. 1.



If we take $\psi = 0$ or 180°,

$$\frac{\delta x}{x} = 3i_1\sin^2 A_1 + 5i_2\sin^4 A_1 + \ldots \quad \delta y = 0\,;$$

if $\psi = 90°$,

$$\frac{\delta x}{x} = i_1\sin^2 A_1 + i_2\sin^4 A_1 + \ldots \quad \delta y = 0.$$

Refraction of Rays not *in one plane.*

If the direction cosines of any ray be $l_0\, m_0\, n_0$ before and $l_2\, m_2\, n_2$ after refraction at the surface (1), the normals being $L_1 M_1 N_1$, we have :

$$\frac{\mu_0 l_0 - \mu_2 l_2}{L_1} = \frac{\mu_0 m_0 - \mu_2 m_2}{M_1} = \frac{\mu_0 n_0 - \mu_2 n_2}{N_1}, \quad \ldots \quad \text{(I.)}$$

where μ_0 and μ_2 are the refractive indices of the two media. Then $\mu_0(l_0M_1-m_0L_1)$ is invariant on refraction.

We may choose the coordinate axes and origin so that

$$L_1r_1,\ M_1r_1,\ \text{and}\ N_1r_1+\beta_1$$

represents the point on the surface at which the refraction takes place. To specify the ray independently of these coordinates we can take the points where the ray cuts the planes $x=0$ and $y=0$.

Let these points be 0, $m_0\rho_0$, d_0; $l_0\rho_0$, 0, D_0, then

$$\rho_0\left(=\frac{d_0-D_0}{n_0}\right)$$

is the distance along the ray between these points.

We have

$$\frac{L_1r_1}{l_0}=\frac{M_1r_1-m_0\rho_0}{m_0}=\frac{N_1r_1+(\beta-d_0)}{n_0}, \quad . \ . \ . \ . \quad \text{(II.)}$$

giving

$$(L_1m_0-l_0M_1)\,r_1=-l_0m_0\rho_0,$$

or $\mu_0l_0m_0\rho_0$ is invariant on refraction.

The condition that L_1r_1, M_1r_1, and $N_1r_1+\beta_1$ represents the point on the surface is satisfied, with different values of r_1 and β_1, for all points on a surface of revolution about the axis of z; and since $\mu_0l_0m_0\rho_0$ is independent of the position of the origin it is *invariant* throughout any system having a common axis.

This invariant may be written $\mu(x_1m-y_1l)$, where x_1 and y_1 are the coordinates of the point in which the ray meets any plane at right angles to the axis.

Considering a complete system we have

$$\mu_0x_0m_0=\mu_2(x_2m_2-y_2l_2),\quad\text{or}$$

$$\mu_0x_0m_0=\mu_2(x_2+\delta x_2)\,m_2-\mu_2\delta y_2l_2,$$

if the ideal values of the coordinates on the image plane be x_2.

Thus

$$\frac{\mu_0x_0m_0}{\mu_2x_2m_2}-1=\mu_2\frac{(m_2\delta x_2-l_2\delta y_2)}{\mu_2x_2m_2}=\frac{\delta x}{x_2}-\frac{l_2}{m_2}\frac{\delta y}{x_2}.$$

It remains to obtain the value of

$$\frac{\delta x}{x_2}-\frac{l_2}{m_2}\frac{\delta y}{x_2}$$

in suitable form.

The aberrations of any centred system may be expressed as

$$\delta x_2 = \frac{\partial}{\partial x_1}\{F(\rho_1^2, \rho_2^2, \chi_{12})\},$$

$$\delta y_2 = \frac{\partial}{\partial y_1}\{F(\rho_1^2, \rho_2^2, \chi_{12})\},$$

where

x_1y_1 represents the point at which the ray meets the stop plane,

x_2y_2 the point in which the ray should meet the theoretical image plane,

$\rho_1^2 = x_1^2 + y_1^2$, $\rho_2^2 = x_2^2 + y_2^2$, $\chi_{12} = +(x_1x_2 + y_1y_2)$,

and F is a function of the second and higher orders in ρ_1^2, ρ_2^2, and χ_{12}.

Then

$$\delta x_2 = 2x_1\frac{\partial F}{\partial \rho_1^2} + x_2\frac{\partial F}{\partial \chi_{12}},$$

$$m_2\delta x_2 - l_2\delta y_2 = 2(m_2x_1 - l_2y_1)\frac{\partial F}{\partial \rho_1^2} + (m_2x_2 - l_2y_2)\frac{\partial F}{\partial \chi_{12}};$$

$$\therefore\ \mu_2(m_2\delta x_2 - l_2\delta y_2) = 2\mu_0x_0m_0\frac{\partial F}{\partial \rho_1^2} + \mu_2m_2x_2\frac{\partial F}{\partial \chi_{12}};$$

$$\therefore\ \frac{\mu_0x_0m_0}{\mu_2x_2m_2} - 1 = 2\frac{\mu_0x_0m_0}{\mu_2x_2m_2}\frac{\partial F}{\partial \rho_1^2} + \frac{\partial F}{\partial \chi_{12}};$$

$$\therefore\ \frac{\mu_0x_0m_0}{\mu_2x_2m_2} - 1 = \frac{2\dfrac{\partial F}{\partial \rho_1^2} + \dfrac{\partial F}{\partial \chi_{12}}}{1 - 2\dfrac{\partial F}{\partial \rho_1^2}}.$$

The denominator is generally nearly unity.

When the object is small and nearly on the axis we need only consider terms of zero and first orders in x_2, y_2, that is,

$$F = f_1(\rho_1^2) + \chi_{12}f_2(\rho_1^2),$$

$$\frac{\partial F}{\partial \chi_{12}} = f_2(\rho_1^2)\ ;\ \frac{\partial F}{\partial \rho_1^2} = \frac{\partial f_1}{\partial \rho_1^2} + \chi_{12}\frac{\partial f_2}{\partial \rho_1^2}.$$

These three terms may be considered separately :—

(1) The lateral spherical aberration is $2\rho_1\dfrac{\partial f_1}{\partial \rho_1}$;

(2) The comatic effects are made up of

$$\delta x = x_2 f_2(\rho_1^2);$$

and (3) $$\delta x = 2x_1\chi_{12}\frac{\partial f_2}{\partial \rho_1^2} = 2x_1^2x_2\frac{\partial f_2}{\partial \rho_1^2};$$

$$\delta y = 2y_1\chi_{12}\frac{\partial f_2}{\partial \rho_1^2} = 2y_1x_1x_2\frac{\partial f_2}{\partial \rho_1^2}.$$

Thus we have

$$\frac{\mu_0 x_0 m_0}{\mu_2 x_2 m_2} - 1 = \frac{\text{lat. spherical}}{\rho_1} + \frac{C_1}{x_2} + \frac{C_2}{x_1},$$

where

C_1 is the x displacement due to Coma when $\chi_{12}=0$.
C_2 is the additional x displacement due to Coma when $\chi_{12}\neq 0$.

Except when x_1 is very small, the last term $\frac{C_2}{x_1}$ becomes unimportant; if x_1 be very small the left-hand side becomes indeterminate and dependent on the actual value of x_2 as chosen.

If we take $\chi_{12}=0$ we have

$$m_2 = \sin A,$$

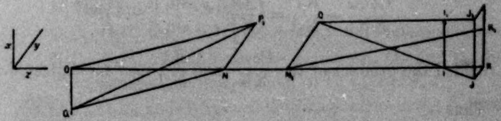
$$\frac{\mu_0 x_0 \sin B}{\mu_2 x_2 \sin A} - 1 = \frac{\text{longit. spher.}}{v_1} + \frac{\text{Coma}}{x_2} = \frac{\delta v}{v} + \frac{\delta x_2}{x_2},$$

and B and A may be obtained from the axial ray if the image be small and v be the distance from the plane (1) to the point at which the axial ray meets the axis.

This result may also be demonstrated as follows :—

Consider the ray OPQI (fig. 2) in the plane yz passing

Fig. 2.



through the system and cutting the axis in I and the theoretical image plane in J, O_1P a neighbouring ray through O_1

where OO_1 is at right angles to the plane yz. The ray OO_1 meets the xz plane in I_1 and the image plane in J_1.

Then

$$\mu_0 \frac{OO_1PN}{O_1P} = \mu_2 \frac{II_1QN_1}{QI_1} \text{ (equation II. above);}$$

but when OO_1 is not large,

$$\frac{PN}{O_1P} = \frac{PN}{OP} = \sin B,$$

$$\frac{QN}{QI_1} = \frac{QN}{QI} = \sin A.$$

Hence

$$II_1 = OO_1 \frac{\mu_0 \sin B}{\mu \sin A}.$$

But if there be spherical aberration present,

$$\frac{JJ_1}{II_1} = \frac{QJ}{QI} = \frac{N_1K_1}{N_1I},$$

and we have

$$JJ_1 = OO_1 \frac{\mu_0 \sin B}{\mu \sin A} \cdot \frac{N_1K_1}{N_1I};$$

if KK_1 be the ideal value of the image $JJ_1 - KK_1 = \text{Coma}$; and we have

$$\frac{\text{Coma}}{KK_1} = \left(\frac{OO_1}{KK_1} \frac{\mu_0 \sin B}{\mu \sin A} \cdot \frac{N_1K}{N_1I} - 1 \right) = \left(\frac{OO_1}{KK_1} \frac{\mu_0 \sin B}{\mu \sin A} - 1 \right)$$

$$+ \frac{KI}{N_1I} \left(\frac{OO_1}{KK_1} = \frac{\mu_0 \sin B}{\mu \sin A} \right),$$

or

$$\frac{\text{Coma}}{KK_1} - \frac{KI}{N_1I} = \left(\frac{OO_1}{KK_1} \frac{\mu_0 \sin B}{\mu \sin A} - 1 \right),$$

provided the product of the $\frac{\text{Coma}}{KK_1}$ and $\frac{KI}{N_1I}$ may be neglected.

Thus

$$\frac{\delta x}{x} + \frac{\delta v}{v} = \left(q \frac{\mu_0 \sin B}{\mu \sin A} - 1 \right). \quad . \quad . \quad . \quad \text{(III.)}$$

Even though the value of II_1 be large, the relation

$$\mu_0 OO_1 \sin PO_1N = \mu II_1 \sin QI_1N_1$$

holds good ; but the point Q is no longer the theoretical image of P or even on the ray OPI.

The relation (II.) above, expressed in the form

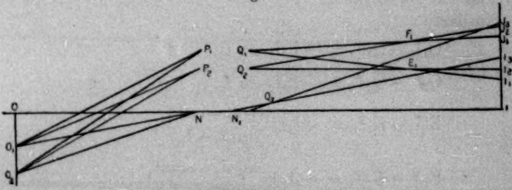
$$\mu \,.\, m \,.\, II_1 \text{ is invariant}$$

furnishes a convenient check upon the results of trigonometrical calculations for oblique rays.

Rays in one Plane.

For the case of rays in one plane we may adapt the method of Hockin to give further information. In this method the optical length, *i. e.*, $\Sigma\mu \times$ actual length, is stationary for the actual path. We cannot, since aberrations are present, assume that *all* optical paths between the same points are of the same length, merely that the difference between two neighbouring rays is zero.

Fig. 3.



All the rays are in one plane.

O_1, O_2 are two neighbouring object points ;

P_1, P_2 two neighbouring points on the stop plane ;

N_1 the centre of the stop ;

O_1P_1, O_1P_2, and O_1N_1 pass through the system and meet O_2P_1, O_2P_2, and O_2N_1 in the points Q_1, Q_2, and Q_3 and intersect the image plane in I_1, I_2, and I_3 ;

O_2P_1, O_2P_2, and O_2N_1 meet the image plane in J_1, J_2, and J_3 ;

Q_1I_1, and Q_2I_2 meet in E_1 ;

Q_1J_1 and Q_2J_2 meet in F_1.

Considering the optical paths of neighbouring rays, we have the optical lengths

$$O_1P_1Q_1E_1 = O_1P_2Q_2E_1,$$
$$O_2P_1Q_1F_1 = O_2P_2Q_2F_1,$$

i. e., $$O_1P_1 - O_2P_1 + (Q_1I_1 - Q_1J_1) - (E_1I_1 - E_1I_2) + (F_1J_1 - F_1J_2)$$
$$= (O_1P_2 - O_2P_2) + (Q_2I_2 - Q_2J_2),$$

or $$(O_1P_1 - O_2P_1) - (O_1P_2 - O_2P_2) + (Q_1I_1 - Q_1J_1) - (Q_2I_2 - Q_2J_2)$$
$$= (E_1I_1 - E_1I_2) - (F_1J_1 - F_1J_2).$$

We denote the angles between the axis and the rays O_1P_1, O_2P_2, O_1N and O_2P_1, O_2P_2, O_2N by $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ respectively; and the angles the axis makes with Q_1I_1, Q_2I_2, Q_3I_3, and Q_1J_1, Q_2J_2, Q_5J_3 by I_1, I_2, I_3 and J_1, J_2, J_3 respectively.

Then

$$O_1P_1 - O_2P_1 = -O_1O_2 \frac{\cos\beta_1 - \cos\alpha_1}{\sin(\alpha_1 - \beta_1)} = -O_1O_2 \sin\alpha_1,$$

provided O_1O_2 be small.

Similarly

$$Q_1I_1 - Q_1J_1 = -I_1J_1 \sin I_1,$$
$$Q_2I_2 - Q_2J_2 = -I_2J_2 \sin I_2,$$
$$E_1I_1 - E_1I_2 = -I_1I_2 \frac{\cos I_2 - \cos I_1}{\sin(I_1 - I_2)},$$
$$F_1J_1 - F_1J_2 = -J_1J_2 \frac{\cos J_2 - \cos J_1}{\sin(J_2 - J_1)}$$
$$= -J_1J \frac{\cos I_2 + Q_2\,(\cos(I_1 + Q_1)\,)}{\sin\{(I_1 - I_2) + (Q_2 - Q_1)\}}.$$

If now we can assume the angles Q_2 and Q_1 small,—that is, I_1J_1 and I_2J_2 small, we have

$$F_1J_1 - F_1J_2 = -J_1J_2 \left\{ \frac{\cos I_2 - \cos I_1}{\sin(I_1 - I_2)} \right\} \left\{ 1 - (Q_2 - Q_1) \frac{\cos(I_2 - I_1)}{\sin(I_1 - I_2)} - \frac{Q_2 \sin I_2 - Q_1 \sin I_1}{\cos I_2 - \cos I_1} \right\}$$
$$= -J_1J_2 \frac{\cos I_2 - \cos I_1}{\sin(I_1 - I_2)} - J_1J_2 \frac{Q_1 \cos I_2 + Q_2 \cos I_1}{(\cos I_2 - \cos I_1)(1 + \cos(I_2 - I_1))}.$$

Thus

$$(E_1I_1-E_1I_2)-(F_1J_1-F_1J_2)=(J_1J_2-I_1I_2)\frac{\cos I_2-\cos I_1}{\sin(I_1-I_2)}$$
$$+J_1J_2\frac{Q_1\cos I_2+Q_2\cos I_1}{1+\cos(I_1-I_2)};$$

or

$$-O_1O_2(\sin\alpha_1-\sin\alpha_2)-(I_1J_1\sin I_1-I_2J_2\sin I_2)=(J_1J_2-I_1I_2)\frac{\cos I_2-\cos I_1}{\sin(I_1-I_2)}$$
$$+J_1J_2\left\{\frac{I_1J_1}{Q_1J_1}+\frac{I_2J_2}{Q_2J_2}\right\}\left(\frac{\cos I_1\cos I_2}{1+\cos(I_1-I_2)}\right).$$

On summing throughout the aperture for O_1 on the axis

$$-O_1O_2\sin\alpha_1-I_1J_1\sin I_1=\Sigma(J_1J_2-I_1I_2)\sin I_1+\Sigma J_1J_2\cos^3 I_1\, 2\,.\frac{IJ}{NI},$$

where the value of J_1J_3 is assumed small as compared with NI.

Thus

$$I_3J_3\sin I_1\left(\frac{O_1O_2\sin\alpha_1}{I_3J_3\sin I_1}-1\right)=-(I_3J_3-I_1J_1)\sin I_1$$
$$+\Sigma(J_1J_2-I_1I_2)\sin I+\Sigma\left(J_1J_2\cos^3 I\,.\frac{IJ}{NI}\right).$$

But $I_1J_1-I_3J_3=$ Comatic error for rays in one plane; and $\Sigma(J_1J_2-I_1I_2)\sin I_1$ is the Coma × the average value of $\sin I_1$: this average value depends on the way in which the Coma varies with the aperture.

If we assume

$$\frac{\delta x}{x}=3i_1\sin^2 A+5i_2\sin^4 A,$$

the mean value in the first term is

$$2i_1\sin^2 A,$$

i. e., $\frac{2}{3}$ of $3i_1\sin^2 A$,

in the second $\frac{4}{5}$ of $5i_1\sin^4 A$.

Thus we have, since $A=-I_1$,

$$\left(\frac{O_1O_2\sin\alpha_1}{I_3J_3\sin A_1}-1\right)=\frac{1}{I_3J_3}\left\{\tfrac{1}{3}\text{ Coma of first order}+\tfrac{1}{5}\text{ Coma of second order}\right\}$$
$$+\frac{J_1J_3}{IN}\frac{1}{\sin I}\Big\}$$
$$=\frac{1}{I_3J_3}\left\{\tfrac{1}{3}\text{ Coma of first order}+\tfrac{1}{5}\text{ Coma of second order}\right\}$$
$$+\frac{\text{long. spher. aberr.}}{IN}$$

This result is equivalent to that proved above for the rays passing through $x_1=0$, on the stop, provided that for $x_1=0$ the Coma may be denoted by

$$i_1 \sin^2 A + i_2 \sin^4 A, \text{ and for } x_1=\rho_1 \text{ by } 3i_1 \sin^2 A + 5i_2 \sin^5 A.$$

This shows that the proper expansion for the Coma is in the form $i_1 \sin^2 A + i_2 \sin^4 A$ rather than $i_1 A^2 + i_2 A^4$, since the latter would not separate the terms which are in the ratio of 3 : 1 from those in the ratio of 5 : 1, and thus would lead to a more complicated expression for the Coma.

The summation above may also be applied to the case of an oblique ray. We then have

$$-O_1O_2(\sin \alpha_1 - \sin \alpha_3) - (I_1J_1 \sin I_1 - J_1J_3 \sin I_3) = \Sigma(J_1J_2 - I_1I_2) \sin I$$
$$+ \Sigma J_1J_2 \cos^3 I \cdot \frac{IJ}{NI},$$

$$i.\,e.\ \left(\frac{-O_1O_2(\sin \alpha_1 - \sin \alpha_3)}{+I_3J_3(\sin I_1 - \sin I_3)} - 1\right) = \frac{1}{I_3J_3(\sin I_1 - \sin I_3)}\Big\{ -(I_1J_1 - I_3J_3) \sin I_1$$
$$+ \Sigma\Big((J_1J_2 - I_1I_2) \sin I_1\Big) + \Sigma\Big((J_1J_2 \cos^2 I \frac{IJ}{NI}\Big)\Big\}.$$

The summation may be effected provided we know the approximate expressions for the quantity J_1J_2, but in the ordinary way these expressions are too complicated to admit of simple expression.

The condition for the absence of Coma for a small object not on the axis is easily found provided one point be perfectly defined :

$$\text{it is } \left(q \frac{\sin \alpha - \sin \alpha_0}{\sin I - \sin I_0} - 1\right) = 0 \text{ throughout the aperture,}$$

where α_0 refers to the ray through the centre of the stop.

In conclusion, it will be seen that the Comatic effects associated with any departure from the sine condition may be calculated, even though spherical aberration be present, from the relation (III.) above.

ABSTRACT.

The condition for the correction of coma in a centred optical system is the well-known Sine condition. This has been proved by Clausius, Helmholtz, Hockin and others, and the importance of this condition in the design of optical systems has been pointed out by Abbe, Steinheil, Conrady, and others. So far as the Author is aware no discussion of the effects of failure to satisfy this condition has been published, and the present paper shows how to obtain the relation between the Coma of a system and the errors in the Sine Condition.

II. *On Cadmium Amalgams and the Weston Normal Cell.*

By F. E. SMITH, *A.R.C.Sc.*[*]

(From the National Physical Laboratory.)

[Plate I.]

WITHIN the past few years a number of researches have been made on the electromotive properties of mercurous sulphate, and as a result the standard cell is now reproducible with a very high degree of accuracy. However, there are many instances of abnormal behaviour of such cells both with regard to constancy of E.M.F. and with respect to change of E.M.F. with varying temperature. The want of constancy has been attributed by Hulett to slow hydrolysis of the mercurous sulphate, and by Steinwehr[9] to a change in the size of the mercurous sulphate crystals. Irregular behaviour with changing temperature has been observed by Tinsley[13], by Janet[15] and Jouast[15,16], and more recently by Cohen and Kruyt[17], and all of these attribute the observed differences to the cadmium amalgam.

Dearlove[1] first showed in 1893 that attention must be paid to the percentage of cadmium in the amalgam, and from 1898 to 1900, Cohen[3], Kerp and Boettger[4], Bijl[6], Puschin[8], and Jaeger[2] made independent investigations on such amalgams.

The research of Bijl's was especially complete and its value appears to have been overlooked in much of the recent work on the standard cell. Possibly the reason is that the electromotive properties of the amalgams were examined at 20°, 25°, 50° and 75° C. only. However, from other data given by Bijl—as pointed out by Cohen and Kruyt—the conclusion may be drawn that a 12½ per cent. amalgam cannot be usefully employed below about 14° C.

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[*] Read October 22, 1909.

Jaeger made observations at room temperature (20° C. approx.) on amalgams varying in content from 1 to 20 per cent. of cadmium, and showed that at that temperature all amalgams containing from 5 to 14 per cent. of cadmium could be used in the anode limb of the Weston cell with the same resulting E.M.F. In a note on Bijl's work Jaeger[7] emphasizes the fact that the cells made by Lindeck[5] and himself with amalgams containing 12 and 13 per cent. of cadmium showed no irregularities when cooled.

No objection appears to have been raised against the 12½ per cent. amalgam until 1905, when the author[10] stated that it was not wise to use such an amalgam at low temperatures (0° C.) but that an 8 per cent. amalgam could be so used. A few experiments were subsequently made with 10 and 12½ per cent. amalgams, but the results were inconsistent. However, an amalgam of less cadmium content than 12½ was preferred for low temperatures and in consequence most of the Weston normal cells at the National Physical Laboratory[12] contain 10 per cent. amalgams. Pending further investigation it was thought wise not to specify the limits of temperature between which a cell containing 12½ per cent. cadmium amalgam might be used, and in preliminary specifications[11] for the standard cell, issued in 1908, no limits of temperature are fixed.

Tinsley[13], in 1908, obtained some results with Weston cells which led him to believe that cells containing 12½ per cent. amalgam should be used with caution below 10° C. However, the cell with 12½ per cent. amalgam had the lower E.M.F. at the lower temperatures, which is not in accordance with the author's observations.

Janet and Jouast[15], [16] obtained results which were opposed to those of Tinsley. They found a 12½ per cent. amalgam cell to be normal at low temperatures (0°·5 C.) and a 10 per cent. amalgam cell to behave abnormally. This conclusion also is directly opposed to our own experience.

Wolff[14] at the National Bureau of Standards, Washington, used 200 Weston cells to determine the relation between E.M.F. and temperature, the range being 0° C. to 40° C. All of these cells were set up with 12½ per. cent. amalgams and only a few behaved abnormally.

In November 1908, Cohen and Kruyt[17] found differences of 0·04, 0·23, 0·13, 0·16, 0·20, and 0·20 millivolt between amalgams of 10 and 12½ per cent. of cadmium when the electrolyte was a solution of cadmium sulphate and the temperature was maintained at 0° C.: the E.M.F.'s of the cells containing the 12½ per cent. amalgams were the greater. Cohen suggests the use of an 8 per cent. amalgam for general use.

The exact constitution of cadmium amalgams is doubtful and is not discussed in the present paper, but there is no doubt that the electromotive properties of the amalgams depend on whether one or two phases are present. When, by increasing the temperature a solid amalgam is converted into a mixture of liquid and solid phases, there is an immediate change in the electromotive temperature coefficient of the amalgam, and a similar change results when the coexisting phases are changed entirely into the liquid state. If the temperature of an amalgam, containing the mixed phases, is raised, the liquid phase is increased and the solid phase diminished; when stable, however, the E.M.F. of an amalgam towards a solution of cadmium sulphate does not depend on the relative amounts of the two phases. Hence, stable amalgams containing different percentages of cadmium, but possessing the two phases, have the same E.M.F. towards a cadmium sulphate solution. With amalgams all solid or all liquid, the E.M.F. varies with the cadmium content.

Bijl[6] determined the percentage composition of amalgams possessing the 2-phase system from experiments on their electromotive properties. His observations were made at 20°, 25°, 50°, and 75° C. Other experiments on the dilatation of the amalgams, rate of cooling, &c., were made at temperatures ranging from −39° to 321°.

The present investigation was commenced in October 1908. The objects in view were:—(1) To trace the cause of the electromotive difference, which sometimes existed and which was sometimes absent, between 10 and 12½ per cent. amalgams. (2) To determine the limits of temperature between which amalgams of various concentrations could be usefully employed. (Requests for information on this point had on several occasions been received from India.) (3) To explain the abnormal E.M.F.'s of cells made by various observers, when the amalgams contained from 12½ to 14 per cent. of cadmium. (4) As the recent International Conference on Electrical Units and Standards (London, 1908) specify a 12½ per cent. amalgam, it was important to decide whether the limits of temperature (0°–40° C.) specified, required amending or not.

The Method.

The method consisted in directly measuring the difference of E.M.F. between two amalgams in contact with a saturated solution of cadmium sulphate. The materials in contact with the amalgams were identical with those used in the Weston normal cell, and any amalgam could at any time be used as the negative element of such a cell.

The glass vessels used were of the form shown in fig. 1. Each vessel has four vertical limbs connected by means of a tubular cross-piece. A platinum wire is sealed through the bottom of each of the limbs and passes into a narrow side-tube containing mercury. A 10 per cent. amalgam was placed in the bottom of one of the limbs, and amalgams of different percentages of cadmium were placed in two of the other limbs; in the remaining limb, mercury and a depolarizer consisting of a paste of mercurous sulphate and cadmium sulphate crystals and solution were introduced. Cadmium sulphate crystals were added so as to form a deep layer over the amalgams and paste, and afterwards the vessel was nearly filled with a neutral saturated solution of cadmium sulphate. The vessel was then hermetically sealed.

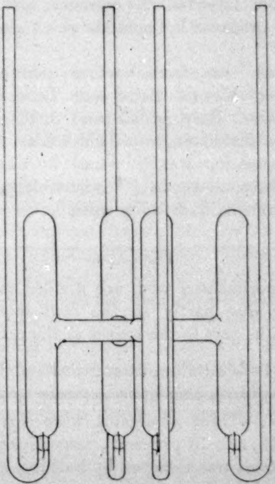
Such a cell will be seen to consist of one positive element and three negative ones, one of which consists of a 10 per cent. amalgam.

The advantages of such a cell are :—

(1) All of the constituents of the Weston normal cell are present.

(2) The electromotive force of any combination having the mercury as the positive element and one of the amalgams as the negative element, may be directly compared with the electromotive force of any standard cell.

Fig. 1.



(3) A large number of the vessels may be grouped together and the positive poles connected. The difference in E.M.F. between any two negative poles, even though they be in different cells, may thus be directly measured.

(4) If there is any doubt about small differences in E.M.F. and a suspicion that they are due to slight differences in the depolarizers, the 10 per cent. amalgam poles

may be connected together, and the differences in E.M.F. measured between these and other amalgams. (Preliminary observations had proved the reliability of the 10 per cent. amalgam.)

(5) The cells may be immersed in ice, water, oil, or other liquid, and they cannot leak through the junction of platinum with glass.

Preparation of the Materials.

The cadmium sulphate, the mercury, and the mercurous sulphate were prepared by methods which are fully described elsewhere[12].

The cadmium was from various sources. Some was purchased from Messrs. Baird and Tatlock; another lot was from Messrs. Harrington's and further samples were procured from Kahlbaum's and Merck's. In addition, a considerable quantity was prepared by electrolysing pure cadmium sulphate, an anode of commercially pure cadmium and a cathode of platinum being used.

The Cells.

The cells experimented with are divided into groups, the division being due partly to the heat treatment of the amalgams, and in part to the source of the cadmium.

Group I.—Cadmium from Baird and Tatlock.

Range of amalgams:—1 to 20 per cent. of cadmium.

Manufacture:—Three amalgams were first made; these contained 20, 15, and 10 per cent., respectively, of cadmium. All oxide and dross was removed by washing with very dilute sulphuric acid while the amalgams were hot and completely liquid. Each amalgam was then chilled with ice-cold water and broken into small pieces. From the 20 per cent. amalgam other amalgams estimated to contain from 19 to 16 per cent. of cadmium were made; the 15 per cent. amalgam was used to prepare others having from 14 to 10½ per cent. of cadmium, and from the 10 per cent. amalgam the 9 to 1 per cent. amalgams were made. The amalgams were introduced into the limbs of the vessels and melted either in a water bath or over a bunsen flame; the cells and amalgams were allowed to cool in a normal manner. Afterwards, the other constituents of the cells were introduced and the vessels were hermetically sealed; they were subsequently immersed in paraffin oil and kept at a constant temperature of 20° C. for 48 hours, after which they were placed in crushed ice for 8 days.

Ten Weston normal cells of the usual H form were continually maintained at a constant temperature of 20° C. and the E.M.F.'s of the various combinations of the elements in the 4 limb vessels were compared with these 10 cells from time to time.

When the experimental cells were at 0° C. the E.M.F.'s were compared every day for 8 days; afterwards further observations at 5°, 10°, and 15° C. were made, but the first two of these temperatures were not maintained constant within less than 0°·1 C., and this for 3 hours only. The temperature of 15° C. was maintained constant for 24 hours before the first comparison of E.M.F.'s was made, and the whole lot of cells did not deviate from 15° C. by more than 0°·5 during the following 3 months. From time to time the temperature was adjusted to 15° C. and the cells compared. After the 3 months had elapsed other comparisons were made at intervals of 5° from 20° to 45°, these higher temperatures being maintained constant for at least 4 hours before making the observations. The oil baths were heated electrically, and a toluene thermostat sensitive to 0°·01 was used. For temperatures above 30° the cells were immersed in crude olive oil.

Results.

The results of the E.M.F. comparisons are given in Tables I and II. Table I. gives the E.M.F.'s of amalgams containing from 1 to 13·5 per cent. of cadmium. When maintained at a constant temperature the E.M.F.'s of these amalgams (with the exception of that containing 6 per cent. of cadmium) were found to be constant also within 1 or 2 parts in 100,000. It is not necessary, therefore, to give more than one value at any one temperature.

TABLE I.—Giving the observed E.M.F.'s at various temperatures of Weston normal cells containing amalgams with from 1 to 13·5 per cent. of cadmium.

Observed E.M.F.'s = 1·01840 int. volts, + the differences given in the Table, the differences being in hundred-thousandths of a volt.

DATE.	TEMP.	Percentage of cadmium in the amalgam.																
		1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	10·5.	11.	11·5.	12.	12·5.	13.	13·5.
28.10.08	0° C.	−1197	−365	+35	34	35	35	36	36	36	36	37	37	39	40	41	47	50
6.11.08	5°	...	−580	−32	35	36	36	37	37	37	37	38	38	40	41	42	48	51
9.11.08	10°	...	−798	−246	29	29	30	30	30	30	30	31	31	33	34	34	35	38
10.2.09	15°	...	−1024	−464	−79	17	17	18	18	18	18	19	19	21	22	22	25	27
15.2.09	20°	...	−1250	−684	−291	−1	−1	0	0	0	0	1	1	3	4	4	4	7
19.2.09	25°	...	−1483	−910	−507	−186	−24	−24	−23	−23	−23	−22	−22	−19	−18	−17	−17	−16
24.2.09	30°	...	−1717	−1132	−729	−407	−140	−50	−49	−49	−49	−48	−48	−44	−44	−43	−43	−42
26.2.09	35°	...	−1959	−1365	−954	−628	−362	−94	−80	−79	−79	−78	−78	−77	−76	−76	−76	−75
1.3.09	40°	...	−2202	−1594	−1184	−853	−583	−315	−114	−114	−112	−111	−112	−112	−112	−112	−111	−111
4.3.09	45°	...	−2446	−1837	−1415	−1081	−804	−539	−296	−148	−146	−145	−145	−146	−145	−145	−144	−144

TABLE II.—Giving the observed E.M.F.'s at various temperatures of Weston normal cells containing amalgams with from 14 to 20 per cent. of cadmium.

Observed E.M.F.'s = 1·01840 v. + the differences given in the Table, the differences being in hundred-thousandths of a volt.

DATE.	TEMP.	Percentage of cadmium in the amalgam.							
		14.	14·5	15.	16.	17.	18.	19.	20.
28.10.08	0° C.	+238	330	696	486	1098	1290	1560	1610
5.11.08	”	249	457	673	560	1101	1295	1564	1613
6.11.08	5° C.	120	329	537	428	969	1162	1419	1476
7.11.08	”	123	339	533	431	969	1163	1420	1476
9.11.08	10° C.	59	205	390	288	830	1022	1270	1332
10.11.08	”	61	215	387	292	830	1023	1270	1333
13.11.08	15° C.	47	89	239	149	683	870	1124	1187
10.2.09	”	63	115	214	244	643	870	1075	1188
11.2.09	20° C.	13	19	55	72	488	717	922	1036
15.2.09	”	13	19	55	70	488	716	922	1035
17.2.09	25° C.	−15	−13	0	0	327	557	760	876
19.2.09	”	−15	−13	0	0	327	555	760	877
22.2.09	30° C.	−42	−41	−31	−40	162	376	594	712
24.2.09	”	−42	−41	−32	−40	159	372	592	710
25.2.09	35° C.	−74	−73	−71	−72	−6	204	423	539
26.2.09	”	−74	−73	−71	−72	−7	197	423	539
1.2.09	40° C.	−111	−110	−108	−109	−108	30	244	361
2.2.09	”	−111	−110	−109	−110	−108	24	241	360
3.2.09	45° C.	−145	−144	−144	−144	−144	−120	+60	170
4.2.09	”	−145	−144	−143	−144	−144	−130	+48	169

With regard to the 6 per cent. amalgam, at 30° C. the following values were observed :—

Date.	E.M.F. = 1·01840 v. +
22.2. 1902.	−0·00059 v.
23.2. „	−0·00127
24.2. „	−0·00140
24.2. „	−0·00140

This change is shown hereafter (p. 35) to be due to the closeness of the 6 per cent. amalgam at 30° C. to the transition temperature (about 28° C.), at which the two phase system changes into a one phase (liquid) system. The transition appears to take place slowly, and the E.M.F. of the cell does not therefore immediately assume a constant value. It is purely accidental that such a change was only observed with the 6 per cent. amalgam.

Many of the cells containing amalgams with from 14 to 20 per cent. of cadmium did not remain constant when the temperature was unchanged. The variations will be seen from Table II., which gives the initial and final values at the different temperatures.

Of the cells dealt with in Table II. the E.M.F.'s of all the cells, with the exception of that containing the 15 per cent. amalgam, increased in value when maintained at 0°. During the 3 months that the cells were at 15° C. the 15, 17, and 19 per cent. amalgam cells fell in E.M.F., and the 14, 14½, and 16 per cent. amalgam cells rose in value. At the higher temperatures there was very little change, but it is worthy of notice that the 18 and 19 per cent. amalgam cells exhibited declining values.

After the cells had remained at 45° C. for 2 days they were again placed in crushed ice and kept there for 6 weeks. The E.M.F.'s were measured from time to time, and were far from constant for the amalgams of higher concentration. The following table (III.) gives the results, and for the sake of comparison the original values at 0° C. are also given.

The cells containing the amalgams with from 2 to 11·5 per cent. of cadmium may be regarded as quite constant. Amalgam cells with from 12 to 13·5 per cent. of cadmium have fallen in E.M.F. by a few hundred-thousandths of a

Table III.

Giving the observed E.M.F.'s at 0° C. of cells containing from 2 to 20 per cent. of cadmium in the amalgams.

Observed E.M.F.'s = 1·01840 int. volts + diffs. in hundred-thousandths of a volt.

Percentage of Cadmium in the Amalgam.	0° C. 5.11.1908.		0° C. 6.2.1909.	0° C. 27.2.09.	0° C. 20.3.09.	Max. difference.
2	−365	An interval of three months during which the cells had been raised to 45° C.	−368	−368	−368	3
3	+35		35	35	34	1
4	34		35	35	35	1
5	35		35	35	35	0
6	35		35	35	36	1
7	36		36	36	36	0
8	36		36	37	36	1
9	36		36	36	37	1
10	36		36	36	37	1
10·5	37		36	36	37	1
11	37		36	36	37	1
11·5	39		38	37	37	2
12	40		36	36	39	4
12·5	41		39	38	38	3
13	47		40	40	40	7
13·5	50		40	38	41	12
14	249		38	49	86	211
14·5	457		48	58	216	409
15	673		443	567	626	230
16	560		163	179	498	397
17	1101		1006	1042	1049	95
18	1295		1260	1266	1266	35
19	1564		1493	1498	1507	71
20	1613		1611	1616	1617	4

volt, while those with amalgams richer still in cadmium (with the exception of the 20 per cent. one) show considerable changes which in two instances (14·5 and 16) amount to 0·004 volt. With increasing time the E.M.F.'s of these cells are observed also to increase, but even after 6 weeks the final values are considerably lower than the original values. The only cell dealt with in Table II. which had practically the same initial and final values at 0° C. (see Table III.) is the 20 per cent. amalgam cell, and this also is the only cell which was practically constant in E.M.F. when any temperature from 0° to 45° C. was maintained constant.

The graphs which result on plotting the values of the percentages of cadmium and the corresponding E.M.F.'s at various temperatures, are shown in fig. 2 (Pl. I.)*. For the purpose of the figure the final values of the E.M.F.'s given in Table II. were chosen. The lower ascending portion of a curve is taken to indicate that the amalgam is completely liquid; the nearly horizontal portion indicates that a two-phase system—part solid and part liquid—is present, and the upper ascending curves show that the amalgam is completely solid.

It is evident, however, from the results given in Tables I. to III. that the electromotive property of an amalgam may depend to some extent on its previous thermal treatment. Whereas, originally, the E.M.F.'s of cells at 0° C. with amalgams containing from 3 to 13·5 per cent of cadmium, were nearly identical, we find after heating to 45° C. and subsequent cooling that the limits are 3 and 14·5 per cent. Similarly at 15° C. an approximate constancy of E.M.F. was originally obtained with 5 to 14 per cent. amalgams, but after the cells were raised to 45° C. and then cooled to 15° the range was extended to 16 per cent. amalgams. *Hence it appears possible that if an amalgam rich in cadmium is raised to a temperature at which the solid and liquid phases are present, it may, on cooling, have different electromotive properties owing to the persistency of the two phases. This persistency may be due to lack of homogeneity of the amalgam.* The curves in fig. 2 show that the amalgams containing from 9 to 17 per cent. of cadmium were partly liquid and partly solid at 45° C., while the 18 and 19 per cent. amalgams were fairly close to their transition temperatures. Referring to Table III. and to the curves in fig. 2, we see that the amalgams which show the greatest changes in their electromotive properties (these are the amalgams containing from 14 to 17 per cent. of cadmium) are partly liquid and partly solid at 45° C., but are normally quite solid at 0° C. Other amalgams containing 14, 15, 16, and 17 per cent. of cadmium were prepared, and their general appearances before and after heating to 45° C. were carefully noted; there is no doubt that these amalgams appeared to be more liquid after the heat treatment. *The final appearance of the amalgams*

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* In some cases the plotted values lie at considerable distances from the corresponding curves.

is explained if we suppose that a layer of amalgam of comparatively low cadmium concentration surrounds a core of higher cadmium concentration.

The curves in fig. 2 are worthy of notice. The slopes of the higher ascending curves are almost identical for the temperatures 15° C. to 45° C., but only the upper part of the higher curve at 0° C. is approximately parallel to the others. (To avoid confusion the curves for 5° and 10° are not given.) Now if the upper part of the curve at 0° is continued so as to be approximately parallel to the curves at 20°, &c., it meets that part of the curve corresponding to the two-phase system at a point indicated by 11 per cent. of cadmium. It is possible that the curve thus drawn was not experimentally realized because of some difference in the thermal treatment of the amalgams, for it has already been shown that the electromotive properties of an amalgam vary with such treatment. If so we conclude that cells with amalgams containing more than 11 per cent. of cadmium *may have* E M.F.'s at 0° C., differing very considerably from the E.M.F.'s of cells with from 3 to 11 per cent. of cadmium. The results also lead one to suggest that the transition temperatures of the amalgams containing from 11·5 to 13·5 per cent. of cadmium are below 20° C. but above 0° C. Experiments to test this point were immediately proceeded with, but before describing them the results obtained with other groups of cells are of interest.

The new groups of cells contain cadmium from Harrington, Kahlbaum, Merck, and from an electrolytic source. Each of the new amalgams was made by adding metallic cadmium to mercury, and all of them were melted inside the glass vessels and allowed to cool in a normal manner. The E.M.F.'s of these cells were compared at many temperatures, but it will be sufficient for the present purpose to give the E.M.F.'s at 0° C. (p. 262). In order that comparisons may be readily made the original E.M.F.'s at 0° C. are also given for the first group of cells.

The results obtained with the four new groups of cells confirm the conclusions arrived at with the cells of Group I., and furthermore suggest that the maximum values for the E.M.F.'s of the cells with amalgams containing high per-

TABLE IV.

Giving the observed E.M.F.'s at 0° C. of various cells containing amalgams with from 1 to 25 per cent. of cadmium.

Observed E.M.F.'s = 1·01840 + the differences given in the table, the differences being in hundred-thousandths of a volt.

Percentage of Cadmium.	GROUP I. Cadmium from Baird and Tatlock.	GROUP II. Electrolytic Cadmium.		GROUP III. Cadmium from Kahlbaum.	GROUP IV. Cadmium from Harrington.	GROUP V. Cadmium from Merck.
		(*a*)	(*b*)			
1	−1197	−1203	−1205	—	—	—
2	−365	−370	− 372	—	—	—
3	+35	+35	+ 34	+34	+34	+34
4	34	35	+ 34	—	—	—
5	35	35	+ 35	34	34	34
6	35	35	35	—	—	—
7	36	35	36	35	34	35
8	36	36	36	—	—	—
9	36	36	36	—	—	—
10	36	37	37	35	34	36
11	37	37	37	35	36	37
12	40	38	37	40	—	—
12·5	41	39	38	38	42	39
13	47	40	38	36	—	—
13·5	50	—	—	41	—	—
14	249	72	38	460	443	378
14·5	457	40	38	519	—	—
15	673	454	65	574	—	—
16	560	451	45	944	938	654
17	1101	78	48	720	—	—
18	1295	455	49	1223	1083	1111
19	1564	1047	866	—	—	—
20	1613	1612	1580	—	1601	—
25	—	—	—	2420	—	2465

N.B.—An interval of 36 hours elapsed between observations (*a*) and (*b*). In this interval the temperature of the cells was raised to 65° C. for 1 hour, and the cells allowed to cool very slowly in the bath. They were then placed in ice and the observations (*b*) taken 6 hours afterwards.

centages of cadmium have not yet been obtained. If, as suggested on p. 260, the result of slowly cooling an amalgam to a temperature at which it should (if homogeneous) be in a solid state is the production of an outer shell of amalgam of comparatively low cadmium concentration, covering a solid core of higher concentration, then different rates of cooling should produce different results. Also, it seems probable that by raising the temperature until the amalgam is completely liquid, and then chilling, the differences of concentration will be largely avoided; but if the amalgam is not in the completely liquid condition at the high temperature, or nearly so, the effect of chilling may not be marked. To test the first effect, a 14 per cent. amalgam was made and divided into two parts; these were placed in two limbs of the same vessel. One of the parts was completely melted and allowed to cool in a water bath down to a temperature of about 16°; the other part was completely melted and chilled by quickly immersing the limb containing it in alcohol cooled with solid CO_2, the temperature being about $-50°$ C. The other constituents of the cell were added, the vessel hermetically sealed and immediately afterwards it was placed in crushed ice. The observed E.M.F.'s at 0° C. were as follows:—

TABLE V.

Giving the observed E.M.F.'s at 0° C. of cells containing 14 per cent. amalgams, one of which was chilled and the other slowly cooled.

The E.M.F.'s = 1·01840 v. + values given in Table.

	14 per cent. slowly cooled.	14 per cent. chilled.
1st day	+0·00038 v.	+0·00606 v.
2nd „	40	656
4th „	40	686
5th „	40	691
6th „	40	693
10th „	41	696
15th „	41	695
21st „	41	694

The results show in a decisive manner that differences of very considerable amounts may arise owing to differences in the rates of cooling of the amalgams. The 14 per cent. slowly cooled amalgam cell had an E.M.F. only 5 parts in

100,000 greater than that of a cell containing a 5 per cent. amalgam, whereas the 14 per cent. chilled amalgam had an E.M.F. which is considerably greater than that of any cell with 14 per cent. amalgam which is reported on in Table IV.

It was thought possible that the chilled and slowly cooled amalgams might differ owing to local differences in concentration other than that mentioned on p. 23. When a 14 per cent. amalgam—or one of higher concentration—is completely melted and slowly cooled in a water bath, the solid crystals which first separate out are lighter than the mother liquor, and rise to the upper portion of the amalgam. Hence, after the amalgam has cooled, it is probable that the concentration of the cadmium in the upper portion is greater than in the lower portion. When a 16 per cent. amalgam was melted in a long glass tube and cooled slowly, this difference was detected by inspection, the lower half of the amalgam having an appearance approaching that of liquid mercury, while the upper portion had a frosted appearance. To test whether there was any marked difference in the electromotive properties of the two ends a platinum wire was sealed through the middle portion of a closed glass tube, 3 mm. in diameter and 30 cm. long ; the tube was then filled with a 16 per cent. amalgam. After the amalgam had been completely melted and slowly cooled in a water bath, the lower end of the tube was broken away so as to expose part of the amalgam's surface. The amalgam rod was now used as the negative element of two cells, the platinum wire serving as the negative lead. In one of the cells the upper part of the surface of the amalgam made contact with cadmium sulphate crystals and solution, and in the other cell the lower part of the amalgam's surface was in contact with a similar mixture. After three hours at 15° C. the cell containing the lower part of the amalgam was the higher in E.M.F. by about 0·0001 volt, but the difference was not constant, and both cells were increasing in value. We conclude, therefore, that the large differences observed between the chilled and slowly cooled amalgams are not due to such local differences of concentration as have been indicated.

Cells with Chilled Amalgams.

Another series of cells were now prepared with amalgams, some of which were melted and afterwards slowly cooled in a water bath, and others were melted and chilled by immersion in alcohol cooled by means of solid CO_2 to about – 50° C. After being chilled the amalgams were not allowed to rise in temperature above 0° C., the cell being prepared with the glass vessel immersed in a mixture of ice and salt. After sealing and swilling with ice-cold water the cells were placed in ice.

With amalgams containing from 1 to 11 per cent. of cadmium no difference greater than two hundred-thousandths of a volt was observed between the electromotive effects of chilled and slowly cooled amalgams of the same percentage composition.

With amalgams of higher concentration the chilled amalgam cell had the higher E.M.F., and its value increased for from 7 to 14 days, after which a slight fall was generally observed. In Table VI. the E.M.F.'s of the various cells are given after the cells had been kept at a constant temperature of 0° C. for 1, 5, 10 and 30 days.

The E.M.F.'s of the cells with the slowly cooled and chilled amalgams—1 to 11 per cent. of cadmium—were practically identical with those given in Table I. (at 0° C.), and so they are not repeated here. However, in order that comparisons may be made with a two-phase system amalgam the E.M.F.'s of the 10 per cent. amalgam cells are given in the Table.

When amalgams of the same cadmium content were melted and chilled, the cells containing them did not always have the same E.M.F. even after several weeks. However, the differences were comparatively small for the amalgams containing more than 14 per cent. of cadmium ; thus after 30 days at 0° C. two cells containing 16 per cent. chilled amalgams had E.M.F.'s of 1·02947 v. and 1·02931 v., and two other cells with 18 per cent. amalgams had E.M.F.'s of 1·03335 v. and 1·03310 v.

In two instances there were very rapid rises in E.M.F. and subsequent rapid falls ; one instance is the 13 per cent. amalgam cell of Table VI., and in the second case a 16 per

TABLE VI.

Giving the observed E.M.F.'s at 0° C. of cells containing slowly cooled and chilled amalgams. Chilled amalgam cells are indicated by "c."

Observed E.M.F.'s = 1·01840 int. volts + the differences given, in hundred-thousandths of a volt.

Percentage of Cadmium in Amalgam.	1st day.	5th day.	10th day.	30th day.
10	36	36	36	36
10 c	36	36	36	36
12	40	39	39	39
12 c	50	183	272	268
12·5	39	39	39	39
12·5 c	41	340	366	360
13	42	44	43	43
13 c	456	650	464	460
14	37	42	42	43
14 c	606	691	695	694
15	44	110	337	544
15 c	583	897	920	910
16	123	592	692	770
16 c	1037	1114	1112	1110
17	50	680	800	872
17 c	1232	1297	1295	1275
18	731	936	988	1017
18 c	1406	1509	1508	1505
19	945	1238	1342	1396
19 c	1601	1672	1676	1668
20	1453	1601	1590	1580
20 c	1770	1840	1840	1826
25	2378	2370	2410	2435
25 c	2496	2529	2550	2530

cent. chilled amalgam cell had an initial E.M.F. of 1·04097 v., which fell in 5 days to 1·03032 v. After 30 days this latter cell had the value 1·02964 v., which is almost identical with the value given for 16 c in Table VI.

Seven cells were prepared with 12½ per cent. amalgam. The first four contained amalgams which had been chilled with alcohol cooled with solid CO_2; the fifth was chilled by immersion in a mixture of ice and salt at $-10°$ C.; the sixth was chilled in water at 0° C., and the seventh was allowed to cool in a normal manner. The E.M.F.'s of the seven cells are given in Table VII.

TABLE VII.

Giving the observed E.M.F.'s at 0° C. of cells containing 12½ per cent. amalgams.

Amalgam.	Observed E.M.F. = 1·01840 +.		
	1st day.	5th day.	30th day.
Chilled to about 50° C.	0·00036	0·00238	0·00222
„ „ „ „	47	273	310
„ „ „ „	41	340	360
„ „ „ „	116	83	80
„ „ „ −10°	52	63	176
„ „ 0°	44	45	44
Cooled normally	40	40	41

The E.M.F.'s of the chilled amalgam cells dealt with in Tables VI. and VII. cannot be regarded as constant even after 30 days. The small decline in E.M.F. from the 10th to the 30th day (Table VI.) was possibly the commencement of a fall which would have lasted a considerable time, and may be due to the chilling producing a too highly concentrated amalgam in the outer shell. However, the uniformity of the results (see fig. 4) leads us to believe that the maximum values of the E.M.F's for homogeneous amalgams are not very different from those observed on the 30th day.

With regard to the slowly cooled amalgams (p. 28), it will be seen that the low initial E.M.F. is explained if the outer shell of the amalgam is of lower cadmium concentration than the mean concentration. With continued diffusion the outer skin of the amalgam will become richer in cadmium, and thus the E.M.F. of the cell will increase. Such increases are recorded in Table VI., and have been previously observed by Jaeger[2] and by Bijl[6].

If, after a cell has been hermetically sealed, a previously slowly cooled amalgam (forming the negative element of the cell) is raised above its first transition temperature, but not above the second, subsequent chilling of the amalgam does not, in general, result in an increased E.M.F., but a lowering may result. A cell containing a 14 per cent. amalgam was raised to 50° C. and then chilled with ice-water ; its E.M.F. was thus lowered from 1·02260 v. to 1·02220 v. This lowering was no doubt due to the outer shell of the amalgam becoming even less concentrated in cadmium than before. However, in such experiments it was repeatedly observed that the rate of increase of E.M.F. of the cell with time was accelerated by such treatment.

After remaining in ice for 30 days, the cells with the chilled and slowly cooled amalgams (Table VI.), and others containing from 2 to 10 per cent. cadmium amalgams, were gradually increased in temperature, and the E.M.F.'s observed at intervals of 5° up to 65°. The temperatures 5° and 10° were maintained very nearly constant for about 1 hour, and all other temperatures were kept constant for at least 4 hours. The results are given in the following Table (p. 31).

After the cells had been raised in temperature to 65° C., they were again immersed in ice, and after 14 hours a comparison of their E.M.F.'s led to the results given in the last column of Table VIII. (compare with Table III.).

With the exception of these final values at 0°, the E.M.F.'s &c., given in Table VIII. have been plotted, and the resulting curves appear in fig. 3 (Pl. I.). The dotted portions of the curves were not experimentally realized, as the cells were unstable at temperatures very near to the first transition points of the amalgams.

A comparison of the effects of temperature changes on chilled and slowly cooled amalgams is now of interest. The whole of the slowly cooled amalgam cells tabulated in Table VI. were raised in temperature from 0° to 65° C., but it is not necessary to give all the results. The following (Table IX.) is a typical case. Here, the three amalgams in one vessel were 10 per cent., 17 per cent. chilled, and 17 per cent. slowly cooled. The temperature was altered comparatively rapidly, the whole range 0°–40° being gone through in a little less than 8 hours.

TABLE VIII.—Giving the observed E.M.F.'s of cells containing previously chilled amalgams.

Observed E.M.F.'s = 1·01840 int. volts + differences given in Table, the differences being in hundred-thousandths of a volt.

Percentage of Cadmium.	0°.	5°.	10°.	15°.	20°.	25°.	30°.	35°.	40°.	45°.	50°.	55°.	60°.	65°.	0° (see p. 259).
2	−365	−580	−798	−1024	−1250	−1483	−1717	−1959	−2202	−2446	...	...	...	...	−366
3	+35	−32	−246	−464	−684	−910	−1132	−1365	−1594	−1837	−2078	−2325	−2574	−2828	+35
4	+34	35	29	−79	−291	−507	−729	−954	−1184	−1415	−1650	−1887	−2131	−2380	+34
5	+35	36	29	17	−1	−186	−407	−628	−853	−1081	−1310	−1543	−1778	−2016	+35
6	+35	36	30	17	−1	−24	−140	−362	−583	−804	−1027	−1250	−1475	−1706	+35
7	+36	37	30	18	0	−24	−50	−94	−315	−539	−760	−985	−1210	−1437	+35
8	+36	37	30	18	0	−23	−49	−80	−114	−296	−521	−748	−971	−1200	+37
9	+36	37	30	18	0	−23	−49	−79	−114	−148	−327	−547	−765	−986	+36
10	+36	37	30	18	0	−23	−49	−79	−112	−146	−181	−350	−565	−786	+36
11	+37	37	31	19	1	−22	−48	−79	−111	−146	−181	−220	−377	−594	+35
12	+268	136	33	19	1	−22	−49	−79	−111	−146	−181	−220	−258	...	+36
12·5	+360	230	86	20	2	−21	−47	−79	−111	−146	−181	−218	−257	−296	+38
13	+460	327	189	45	3	−21	−47	−78	−112	−146	−181	−217	−258	−297	+41
14	+694	560	422	261	105	−10	−46	−77	−110	−145	−181	−217	−256	−296	+42
15	+910	778	638	487	330	176	21	−76	−109	−146	−181	−217	−255	−295	+127
16	+1110	985	846	700	543	383	218	61	−89	−144	−180	−217	−254	−294	+364
17	+1275	1143	998	853	700	542	377	208	35	−106	−176	−217	−247	−290	+667
18	+1505	1373	1232	1085	925	763	602	430	251	80	−70	−197	−247	−290	+601
19	+1668	1532	1390	1244	1083	924	760	589	409	223	35	−70	−230	−291	+850
20	+1826	1692	1553	1406	1250	1076	909	739	559	375	188	20	−160	−290	+1517
25	+2530	2398	2257	2111	1950	1787	1625	1450	1277	1091	900	710	511	321	+2516

TABLE IX.

Observed E.M.F.'s = 1·01840 + the differences in hundred-thousandths of a volt, given in the Table.

Date.	Time.	Temp.	Amalgam.		
			17 %. Slowly cooled.	17c. %. Chilled.	10 %.
20.5.09	9.30 A M.	0°	872	1230	36
	10.30	14·52	467	825	21
	11.30	20·0	294	660	0
	12.30 P.M.	30·0	22	335	−49
	1.0	30·0	6	333	−49
	2.0	30·0	−8	333	−49
	3.0	35·0	−72	165	−79
	4.0	36·0	−82	133	−83
	5.0	0°	+196	1205	35
30.5.09	9.0 A.M.	0°	795	1222	36
	1.0 P.M.	20·0	215	650	0
	3.0	25·0	51	500	−23
	4.30	0°	791	1218	34
31.5.09	9.0 A.M.	0°	792	1219	36
	12.30 P.M.	40°	−104	70	−111
	2.0	45°	−142	−140	−146
	4.0	50°	−178	−182	−181
	5.0	0°	37	433	33
1.6.09		0°	540	938	36
10.6.09		0°	850	1088	36
10.7.09		0°	914	1113	36

It will be observed that the E.M.F.'s of the cells with the amalgams 17 and 17c changed with the rise of temperature from 0° to 20° by almost identical amounts. At 30° the E.M.F. of the cell with the slowly cooled amalgam was near to that of the 10 per cent. amalgam cell, and with increasing time, the temperature remaining constant, the two E.M.F.'s got nearer and nearer together. On cooling to 0° a marked change was evident, the cell with the slowly cooled amalgam having fallen by 6·7 millivolts. In 10 days the E.M.F. had risen again within 0·8 millivolt of the original value, and no further change of importance was produced on warming the cell to 25° and subsequent cooling to 0°. During the whole period the change in E.M.F. of the chilled amalgam cell did not exceed 0·2 millivolt. Both amalgams changed considerably on raising the temperature to 50° C. and subsequent cooling, but although at 50° both cells had nearly the same E.M.F., they differed very considerably at 0°. In all cases cells containing amalgams in a presumably solid state, whether chilled or not, could be raised in temperature and subsequently cooled without appreciable change in E.M.F., unless with increasing temperature there was an abrupt change in the temperature coefficient. In such a case the temperature coefficient appears to diminish at first, but afterwards the E.M.F. of the cell rapidly falls to a value which is nearly identical with that of a cell containing a stable two-phase amalgam. In our experiments the change usually occurred when the E.M.F. was from 0·4 to 2·5 millivolts greater than that of a normal cell. In all cases subsequent cooling showed that a change in the amalgam had taken place as the new E.M.F. was less than the previous value at the same temperature.

At temperatures near to but below the first transition temperature the diffusive processes in an unstable amalgam are no doubt accelerated, and the outer shell becomes richer in cadmium with a corresponding increase in the E.M.F. of the cell; even so the diffusion is very slow, and in general before the first transition temperature is reached the outer shell becomes partly liquid, and a two-phase system appears at a lower temperature than would be the case if the amalgam were homogeneous. On subsequent cooling a new outer shell, with even less cadmium than before, is formed, but the inward concentration gradient is greater than before, and hence the cell is more unstable. The results given in Table IX. are easily explained on such an hypothesis.

If a stable amalgam is taken in which the solid and liquid phases are certainly present, increase of temperature will diminish the solid and increase the liquid phase. When the liquid phase alone is present the temp. coef. of a cell containing the amalgam is very different from that of a cell with a two phase system. This is evident from fig. 3. It was surprising to find that even when the temperature of a cell was maintained several degrees above the transition temperature (corresponding to the change to the all liquid phase), several hours elapsed before the transformation was complete. For example, the 8 per cent. amalgam appears to be completely liquid at all temperatures above 41° C., yet when a cell containing such an amalgam was kept at 45° for four hours its E.M.F. was in exact agreement with that of a cell containing a 10 per cent. amalgam, of which the transition point is about 51°. At the end of five hours the E.M.F. began to fall, and at the end of two more hours it became constant but 1½ millivolt lower than previously. This liquid amalgam cell was now perfectly stable and could be raised in temperature to 65° or lowered to 41° without any permanent change such as was observed with the solid amalgams (see also data, following Table I., with regard to a 6 per cent. amalgam). Probably the reason of this behaviour lies in a difference of concentration of cadmium in the amalgam in the upper and lower parts, such as was indicated on p. 26. In a two-phase system, since the solid crystals are lighter than the mother liquor they naturally rise to the upper part of the amalgam, and it is this part which is in contact with the cadmium sulphate solution. When the transition temperature corresponding to the amalgam is reached, the lower half will be unsaturated with cadmium and therefore completely liquid, while the upper part will contain an excess of cadmium and be part solid and part liquid. If the temperature is raised the E.M.F. of the cell will correspond to one with a two-phase system until the upper part of the amalgam is completely liquid.

We will now fix certain limits of temperature, between which various amalgams may be most usefully employed in the standard cell. Fig. 3 (Pl. I.) gives the desired information and the results are tabulated below.

These values have been plotted and give the curves shown in fig. 4 (Pl. I.).

TABLE X.

Percentage of Cadmium in the Amalgam.	Limits of Temperature for use in Weston Normal Cell.	
	Lower Limit.	Upper Limit.
3	below 0° C.	about 3°·4
4	"	" 12°·7
5	"	" 21°·1
6	"	" 27°·7
7	"	" 34°·6
8	"	" 41°·0
9	"	" 46°·0
10	"	" 51°·0
11	about 0° C.	" 56°·0
12	8°·7	" 60°·0
12·5	12°·1	above 60°
13	16°·1	"
14	24°·0	"
15	32°·5	"
16	39°·7	"
17	45°·3	"
18	52°·6	"
19	57°·0	"
20	62°·1	"

Effect of Temperature Changes on the Weston Normal Cell.

It must not be concluded from the curves shown in fig. 3 that the change in the E.M.F. of a two-phase amalgam towards a cadmium sulphate solution has a very small temperature coefficient. The curves indicate the change in E.M.F. with change of temperature of a Weston normal cell; that is, the difference between the change of E.M.F. of mercury towards a cadmium sulphate solution (plus mercurous sulphate), and the change of E.M.F. of a cadmium amalgam towards the solution.

Because of its small temperature coefficient the Weston cell is sometimes used in an unprotected state with respect to temperature changes, and instances have been brought to notice in which one limb of a cell has been 3° higher in temperature than the other limb. This produced a change in E.M.F. of nearly 1 part in 1000. Rapid changes in the temperature of a cell are of little importance if both limbs are at the same temperature.

To determine the temperature coefficient of each limb we have used H form vessels in which the two limbs are 8 inches apart. The tube connecting the limbs is bent in the form of U near to one end; this construction prevents diffusion of the hot liquid from one limb to the cold liquid of the other. Four such cells were made and their electrodes consisted of

(1)	Cadmium 10 %.	Cadmium 10 %.
(2)	„ „	Hg_2SO_4 and Mercury.
(3)	„ „	„ „
(4)	Mercury and Hg_2SO_4	„ „

At the beginning the cells were placed in ice and their E.M.F.'s determined. The one limb was then placed in a bath which could be raised in temperature, while the other limb was kept at 0°. The results are contained in the following table and show the importance of keeping a cell screened, in order that the difference of temperature of the two limbs shall not be appreciable.

The difference between the two E.M.F.'s at the same temperature is almost identical with the difference calculated by the use of the formula given by Wolff[14].

TABLE XI.

Amalgam cell no. (1). Hot limb +. E.M.F.'s at various temperatures.			Mercury cell no. (4). Hot limb +. E.M.F.'s at various temperatures.		
Cold limb.	Hot limb.		Cold limb.	Hot limb.	
0°	0°	0·00000 v.	0°	0°	0·00000 v.
0°	5°	144	0°	5°	145
0°	10°	297	0°	10°	291
0°	15°	459	0°	15°	440
0°	20°	630	0°	20°	593
0°	25°	809	0°	25°	750
0°	30°	996	0°	30°	907

Comparison of Results.

A comparison of some of our results with those of other observers is of interest.

Jaeger made measurements on various amalgams at about 20° C.[2] and at this temperature, cells containing amalgams with from about 5 to 14 per cent. of cadmium were found to give nearly identical results. A 20 per cent. amalgam cell gradually increased in E.M.F. until the latter was 11·5 millivolts greater than the E.M.F. of normal cells. This difference is not far removed from the difference 12·5 millivolts for the 20 per cent. chilled amalgam (Table VIII.), and the difference 10·4 millivolts for the slowly cooled amalgam (Table II.). Jaeger made no observations at 0° C.

Bijl's[6] experiments on the electromotive properties of cadmium amalgams were made at 20°, 25°, 50° and 75°. From the published results it appears that at 25° amalgams containing from about 5·8 per cent. (=9·9 atomic per cent.) to 15·7 per cent. (=25 atomic per cent.) of cadmium possess the two-phase system. From fig. 4 it will be seen that our own experiments fix the limits at 25° as 5·6 and 14 per cent. The lower limits may be taken as identical.

At 50° we conclude from curves given by Bijl that the limits are about 9·9 per cent. (16·4 atomic per cent.) and 20 per cent. (30·8 atomic per cent.) of cadmium, whereas fig. 4 of this paper gives the limits at 50° as 9·7 per cent. and 17·7 per cent. Again the lower limits may be taken as identical. Probably, the upper limits are different because the amalgams used by Bijl were not chilled. Bijl remarks that the quite solid amalgams reached their final values within a few days except in the case of those amalgams which lay in the neighbourhood of the end of the nearly horizontal portion of the curve (see fig. 2 of this communication), *i. e.* the amalgams which were near the transition point. He states also that on passing to lower temperatures these mixtures will certainly become completely solid with a corresponding change in the E.M.F. of the cell. He repeatedly noticed that in such a case the E.M.F. attains its final value very slowly, and ascribes this to the last quantity of liquid amalgam (the surface being solid) requiring a long time to thoroughly mix with the other portion. No observations on the electromotive properties were made at 0°, but one or two observations on the melting-points of the amalgams were made in the neighbourhood of 0°. From these latter observations we may infer (as recently pointed out by Cohen[17]) that the amalgams with from 2·4 per cent. to about 9 per cent. of cadmium may be usefully employed at 0°. Our own results fix the limits as 2·6 and 11 per cent. As one of the curves given by Bijl was obtained by extrapolation below 8° the agreement may be considered satisfactory.

Cohen and Kruyt's[17] observations at 0° C. were made with 10 and 12½ per cent. cadmium amalgams. At 0° the maximum observed difference between cells containing the amalgams was 0·25 millivolt. With chilled amalgams our own observations show that the difference may be more than 10 times as great, the difference observed being 3·30 millivolt. Cohen also measured differences between two 10 per cent. amalgams at 0° C., but the differences were sometimes positive and at other times negative.

Conclusions.

1. Cadmium amalgams of such a composition that if homogeneous * they would be completely solid below certain temperatures, may not in a Weston normal cell have that E.M.F. towards a cadmium sulphate solution corresponding to such a solid. The general result is a lowering of the E.M.F. of the cell and is due to lack of homogeneity of the amalgam, the outer shell of which is of low cadmium concentration and the centre of the mass of high cadmium concentration. Diffusion tends to restore uniformity, and in consequence the cell is unstable for a very considerable time—the E.M.F. rising.

2. When the amalgam is at a temperature near to, but below, the first transition point, the difference of concentration between the inner and outer parts of the amalgam need be only small to enable the outer shell to be a two-phase system. Owing to the small difference of cadmium concentration throughout the mass the diffusion process will be slow

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* The term is used in a relative sense.

and the E.M.F. of the cell may remain constant for a very long time. There is, however, in general, a small difference in E.M.F. between such a cell and one containing a normal two-phase system.

3. Cadmium amalgams of such a composition that, if homogeneous, they would be all liquid above certain temperatures, may not in a standard cell have that E.M.F. towards the solution corresponding to a homogeneous amalgam. In such a case the E.M.F. of the cell is higher than usual. The effect is due to the upper part of the amalgam being a two-phase system and the lower part an all liquid system of less cadmium concentration than the upper part. Diffusion quickly equalises the concentration and the E.M.F. falls to a normal value.

4. The 12½ per cent. cadmium amalgam at present used in the Weston normal cell may be completely solid at temperatures below 12°, and cells containing it may have a higher E.M.F. than normal if used below that temperature. Fortunately, even at temperatures such as 0° C. the amalgam is comparatively near to its first transition point, and the conditions are similar to those noted in (2). In consequence, except in rare cases, cells containing the 12½ per cent. amalgam will behave at 0° as though they contained an amalgam which was always in a two-phase system at that temperature.

5. As a 10 per cent. amalgam at 0° is above its first transition temperature this may be used in standard cells at low temperatures. Such cells may also be used at temperatures as high as 51°. We suggest that the 12½ per cent. amalgam in the Weston normal cell be replaced by a 10 per cent. amalgam.

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ABSTRACT.

Cadmium amalgams may be solid, liquid, or a mixture of solid and liquid phases, the percentage composition of the phases depending on the temperature. When a completely liquid amalgam is cooled below the lower transition temperature, the centre of the resulting solid is of high cadmium concentration, and the outer skin of low cadmium concentration. Diffusion tends to produce uniformity, and in consequence the e.m.f. of a cell containing the amalgam is unstable for a considerable length of time. When the amalgam is slowly cooled to a temperature a little below the lower transition temperature, the difference of concentration between the inner and outer parts of the amalgam need be only small to enable the outer skin to be a two-phase system. The diffusion process will be slow and the e.m.f. may remain constant for a very long time. Amalgams which were of uniform cadmium concentration throughout were obtained by chilling completely liquid amalgams to a temperature of about −50° C.; although not initially stable, rapid diffusion processes resulted in these amalgams becoming approximately uniform throughout after a few days, and their electromotive properties were markedly different from those of slowly cooled amalgams. If, however, the temperature of such an amalgam is raised until two phases exist, subsequent cooling will not restore the uniform condition and an unstable amalgam results. The experiments indicate that a 12·5 per cent. amalgam may be safely used at all temperatures between 12° and 60° C., and a 10 per cent. amalgam at all temperatures between 0° and 51° C. Experiments were made on the temperature coefficients of the anode and cathode limbs of the Weston normal cell, and show that if a difference of temperature of 1° C. exists an error of about 3 parts in 10,000 is introduced.

Discussion.

Mr. S. W. J. Smith expressed his admiration of the paper as a further example of the Author's great experimental skill. The Author had confined himself almost entirely to a careful and detailed account of his experiments, but it seemed that the results would have been easier to assimilate and more interesting (although not more important) if some further attempt had been made to supply a theoretical basis. So far as he (the speaker) could judge, many of the results could be explained very simply by means of well-known generalizations fixing the conditions of equilibrium in systems such as those with which the Author had worked. The results corroborated (and exhibited the value of) the scheme of equilibrium for cadmium amalgams first proposed by Roozeboom in connection with the experiments performed under his direction by Bijl. Their discussion from this point of view would occupy more time than was at his disposal and could be postponed. It might serve a useful purpose, however, if he were to ask for a more explicit opinion on one point. In a series of amalgams (of different percentage compositions) yielding the same e.m.f. at a given temperature, he (the speaker) did not know whether it was necessary, in order to avoid electrolytic effects, that the whole surface of each amalgam should be fluid. In the stable two-phase amalgams, it might be that (so far as electrolytic action was concerned) the solid and liquid phases could co-exist in contact with the solution. In unstable amalgams, cooled rapidly to the temperature of experiment, containing a greater percentage of cadmium than either component of the two-phase system and yet partially fluid, he thought the tendency of electrolytic action (as of diffusion) would be to reduce the amount of fluid amalgam in the surface-layer.

In reply Mr. F. E. Smith said he was glad to know that the results were in agreement with phenomena common to alloys of the binary system. The difficulty of obtaining perfectly stable amalgams necessitated caution in the interpretation of the results, and in many instances, when, according to the theory of alloys, there should have been perfect agreement in the electromotive properties of a series of amalgams, small but decided differences had been observed. With increasing time these differences might disappear.

With regard to the state of the anode surface, he did not wish to imply that in stable two-phase amalgams the whole of the surface was in a fluid condition, but with unstable amalgams which were part solid and part liquid, he thought the evidence was in favour of such a surface.

III. *The Relation between Uranium and Radium.*—IV.*

By FREDERICK SODDY, *M.A.*

THE measurements of the amount of radium in the uranium solutions purified and prepared by Mr. T. D. Mackenzie have now been in progress for nearly four years, and during the past year have all begun to yield certain evidence of the growth of radium. It may be recalled that the quantity of radium in the oldest preparation, which contained 255 grams of uranium, remained almost constant within the error of measurement for the first two years, while in the next year a very slight increase was noticed. It was deduced from this result that the period of average life of the intermediate parent of radium (ionium) must not be less than 16,500 years, on the assumption that, neglecting uranium X, it was the *only* intermediate body in the disintegration series. During the fourth year, now nearly passed, the measurements have been taken to a higher degree of accuracy than formerly, various improvements having been effected, and these have established that the production of radium within this period has been going on at a rate proportional to the square of the time, within the error of measurement. As the methods employed are the outcome of considerable experience, and as the details are of importance in securing accuracy, it may be useful to give a short account of them as they are now carried out, though they do not involve anything new in general principle. Throughout the term "unit leak" will be used to denote a leak of 1 division a minute in the electroscope, and the term "unit of radium" the quantity 10^{-12} gram. One division of the eyepiece-scale of the microscope equals ·023 mm.

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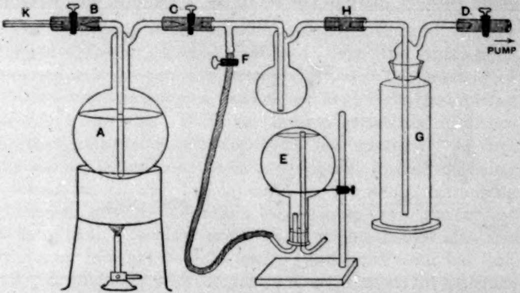
* I. Phil. Mag. June 1905, p. 768; II. (in conjunction with T. D. Mackenzie, B.Sc.), *ibid.* Aug. 1907, p. 272; III. *ibid.* Oct. 1908, p. 634,

A preliminary account of some of the results in this paper appeared in a letter to 'Nature,' May 13th, 1909, p. 308, and the *Physikalische Zeitschrift*, 1909, p. 396. The paper was read on Oct. 22nd, 1909, before the Physical Society of London, under the title "The Production of Radium from Uranium."

2. *Method of Testing for Radium.*

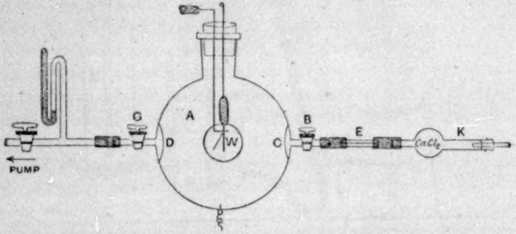
The uranium preparations at the commencement of the fourth year were removed from the mercury-pump, to which they had previously been connected, and transferred to large flasks, as shown at A, fig. 1.

Fig. 1.



In the new method of testing the principle is the same as was first employed by Boltwood. After each measurement the flask A is sealed at B and C, after being exhausted through C, and is left for at least a month for the equilibrium amount of emanation to accumulate. The apparatus as shown in fig. 1 does not need detailed description. Before the measurement it is exhausted through D, and then water in the flask E is admitted through F, by opening the clamp. This is repeated several times to remove any emanation in the water. The flask A is heated gradually by steam on the water-bath for a short time before the seal is broken to avoid percussive boiling. Violent irregular boiling is necessary thoroughly to agitate the liquid and to remove the emanation. If the flask is immersed in warm water boiling tends to be confined to the upper layer, and the extraction of emanation is imperfect. The seals of the flask are broken under thick rubber tubing, a little fresh air being allowed to enter occasionally as required through the thermometer-tube K. After half an hour of vigorous boiling, the clamp C is closed and F opened, admitting water until all but a small definite volume of G is filled. H is then disconnected, and some air thus allowed to enter, removing nitric oxide if present. The tube F is then connected to G at H, and the arrangement is ready for the transference of the emanation into the electroscope. The latter is shown in fig. 2. It consists of a glass globe A of about half a litre capacity, provided with a circular plane parallel plate W to serve as a window, and silvered internally except opposite the window. The window greatly improves the sharpness of the image in the microscope, and the accuracy of the readings. C, D are circles of aluminium-foil to act as baffle-plates, which are pasted over the inside at the inlet and outlet taps B and G. The fine thermometer-tube E also assists in moderating the disturbance of the leaf by the entering air. The electroscope is kept exhausted and charged when not in use during any series of successive measurements. Its normal natural leak has been constant within 10 per cent. at 1·1 units since it was constructed. This is first determined accurately. Several small quantities of air are admitted and pumped out, before finally filling the instrument with air. The natural leak is then taken over several hours, prior to the admission of the emanation. When this has been done the instrument is exhausted and the calcium-chloride tube K connected to the tube D of fig. 1. When the whole of the gas in G has been drawn in, D is disconnected and fresh air allowed to enter to atmo-

Fig. 2.



spheric pressure. No trace of emanation is thus lost. The leak is taken over the period from 8 to 16 minutes after the admission of the emanation, and again after the lapse of three hours. The leaf is kept continuously charged (negatively) with the charging-rod earthed over the whole interval. This is important, as otherwise a considerable error is introduced, which does not appear to have been previously pointed out. The error just referred to applies particularly to the calibration of the electroscope. With the minute quantities of radium present in the uranium solutions, the leaf if fully charged initially remains charged over the three-hour interval, and the practice throughout since this method of measurement was adopted has been to keep it charged during the whole interval. With the radium standards, however, the charge has to be renewed every few minutes. At first this was not done, and the active deposit, instead of being concentrated on the leaf system in the most favourable position to produce ionization, is distributed over the surface of the globe. The leak under these circumstances after the three-hour interval is from ten to twenty per cent. too low, and rises gradually with successive measurements for a further period of three hours.

3. *Calibration of the Electroscope.*

The two standards used in previous work had been made by directly weighing 2·9 and 3·5 milligrams of pitchblende of 53 per cent. uranium content, and differed from each other by about ten per cent., due no doubt to the error of weighing directly such small quantities on an ordinary balance. For the present electroscope an entirely new set of six standards were prepared with the utmost care, from a different sample of pitchblende. Three separate quantities weighing 59·45, 133·7, and 58·32 milligrams were separately dissolved in nitric acid which had recently been distilled over barium nitrate to remove any sulphuric acid, and diluted in 100 c.c. graduated flasks. Out of each of the three solutions two standards were made, by taking a known fraction of the liquid, determined by weighing, and sealing it up after dilution in a flash similar to A (fig. 1) of about

100 c.c. capacity. These six new standards agree extremely well among themselves. On the remainders of the solutions used in preparing the standards three analyses of uranium were performed, which agreed in showing that the proportion of uranium in the pitchblende used was 49 per cent. The results are shown in Table I.

TABLE I.

Standard.	I. PITCHBLENDE. (Milligrams).	II. URANIUM. (Milligrams).	III. RADIUM. ($\times 10^{-12}$ gram.)	IV. LEAK. (Divisions per minute).	V. CONSTANT. (III. ÷ IV.).
I....	0·628	0·308	104·6	18·0 18·1	5·80
II....	3·040	1·490	506·4	86·5	5·85
III....	1·420	0·696	236·6	41·6	5·69
IV....	13·59	6·660	2226·6	(370)	(6·02)
V....	0·305	0·149	50·8	8·6	5·91
VI....	1·510	0·740	251·6	44·5	5·66

MEAN (omitting IV.) ... 5·78

The factor used to calculate the quantity of radium from the quantity of uranium is $3{\cdot}4 \times 10^{-7}$, instead of $3{\cdot}8 \times 10^{-7}$, as previously used. It will be seen that the standards differ among themselves in content of radium in the ratio of over 40 to 1, but the constant of the instrument as determined by the individual standards does not vary from the main value more than two per cent. It must be mentioned that the value for standards IV. is not comparable with the others. The quantity of radium was too high to permit of the charge on the leaf being maintained as it leaked away, so the leaf was kept permanently connected through the charging-rod to the negative pole of the 250 volts mains. Under these conditions the charging-rod as well as the leaf-system would attract the active deposit, and the part deposited near the cork in the neck of the flask would not contribute its full effect. For this reason the constant, as is to be expected, is too high, and the value has been omitted in calculating the mean. The latter is 5·78, that is $5{\cdot}78 \times 10^{-12}$ g. of radium produces a leak of 1 division a minute. The two old standards give a mean value of 7·3 for the constant, the leaf not being kept charged during the three-hours interval. This difference is not much, if any, greater than can be accounted for by the effect of keeping the leaf charged.

4. *Results.*

The results are shown in Table II. for the three uranium preparations described under Experiments I., II., and IV. (II. pp. 285–289 and 290). The later tests in each prepara-

TABLE II.

Date.	Time (years)	$(\text{Time})^2$.	Observed Leak.	Reduction Factor.	Reduced Leak.	Radium ($\times 10^{-12}$ gram).
EXPERIMENT I. 255 grams Uranium purified, 24/10/05.						
9/6/08	2·62	6·86	2·89	×·96×1·15	3·2	18·5*
7/8/08	2·78	7·70	3·3	×·96	3·17	18·4*
25/9/08	2·95	8·57	3·53	×·96	3·39	19·6
20/11/08	3·08	9·49	3·44	...	3·44	19·8
5/5/09	3·53	12·46	3·90	...	3·90	22·5
14/6/09	3·64	13·24	4·20	...	4·20	24·3
27/8/09	3·85	14·80	4·28	...	4·28	24·7
29/9/09	3·93	15·43	4·16	...	4·16	24·1
EXPERIMENT IV. 278 grams Uranium purified, 14/8/06.						
30/5/08	1·80	3·24	2·73	×·96×1·15	3·02	17·5
7/5/09	2·73	7·45	3·68	...	3·68	21·3
25/6/09	2·87	8·22	3·72	...	3·72	21·5
4/10/09	3·15	9·90	4·00	...	4·00	23·1
EXPERIMENT II. 408 grams Uranium purified, 13/12/06.						
8/8/08	1·66	2·75	0·70	×·96	0·67	3·87*
19/11/08	1·93	3·73	0·80	×·96	0·77	4·45
6/5/09	2·40	5·76	1·25	...	1·25	7·20
15/6/09	2·50	6·25	1·26	...	1·26	7·30
1/10/09	2·80	7·83	1·37	...	1·37	7·90

tion have been performed with the electroscope in its existing condition, for which the constant has been accurately determined as described. The first tests were done with another leaf-system, which was destroyed after one test owing to nitric oxide having been present in the gases boiled out of the solution. The factor necessary to reduce the readings with this system to the same value as those with the present system was found to be 0·96. In the initial test shown both for Experiments I. and II., to which not much weight can be attached, the charging-rod was left after charging close to the system as in all previous measurements, whereas the present practice is to turn it as far as possible away, which was found to increase the sensitiveness 1·15 times. The three tests indicated by a * in the last column of the table were done by the mercury-pump before the solutions had been transferred to the sealed flasks they are now kept in. The table shows that in all three preparations the quantity of radium has increased with time. The measurements are plotted against the square of the time in fig. 3. It will be seen that all the points lie close to straight lines. The maximum departure is in the case of the last test performed in Experiment I., which lies about 2 units too low. The

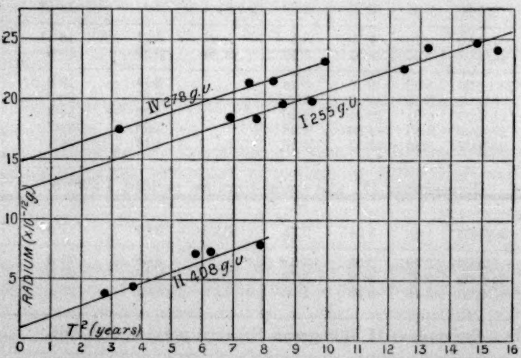


Fig. 3.

ordinary error is not more than 1 unit, or 10^{-12} gram. Considering the nature of the measurement the agreement is perhaps closer than might have been expected, as it is doubtful if any previous measurements of these small quantities of radium have been made to anything approaching this degree of accuracy.

V. *Period of the Direct Parent of Radium.*

As detailed in the last communication (III. p. 635) the growth of radium from uranium should proceed according to the same power of the time as there are products in the series (counting radium itself) between the two elements, provided all these products have periods long compared with the time of observation. Rutherford, assuming only one intermediate product—the direct parent of radium—to exist, first showed that the production of radium should proceed according to the formula

$$R = 1/2\,\lambda_2\lambda_3 R_0 t^2 \quad \text{or} \quad 1/2\,\lambda_1\lambda_2 U t^2,$$

where R represents the quantity of radium, U that of uranium, R_0 the quantity of radium in radioactive equilibrium with the uranium, λ_1, λ_2, λ_3 the constants of uranium, the direct parent of radium, and radium respectively, and t is the time. This may be written

$$R = 6 \times 10^{-8}\lambda_2 t^2,$$

where R represents the radium formed per kilogram of uranium and λ_2 and t are expressed in years. It is of interest to calculate the value of λ_2 from this equation, although, as will be shown later, the calculation is seriously in error if other intermediate bodies of period comparable with the time of observation exist in the series From the curve (fig. 3) representing Experiment I. the growth in 4 years ($t^2=16$) is about 13·3 units, or 52 units per kilogram of uranium. The value of λ_2 is therefore $5{\cdot}4 \times 10^{-5}$, and the value of $1/\lambda_2$, the period of average life, is **18,500 years.** The lower value (10,000 years) given in the preliminary communications is solely due to a value having been used for the constant of the instrument, which is subject to three corrections all operating in the same direction : (1) the effect of keeping the leaf charged, (2) the effect of altering the position of the charging rod, and (3) the alteration in the value of the ratio between U and Ra in pitchblende.

In Experiment IV. the value depends entirely on the initial observation to which, as already mentioned, not much weight can be attached. The experiment shows for the first three years a growth of 7·65 units, or 27·5 units per kilogram of uranium. This gives for λ_2 a value $5·1 \times 10^{-5}$, and for $1/\lambda_2$ 19,600 years.

Experiment III. is very interesting on account of the very small total amount of radium present, though the quantity of uranium is considerably greater than in the other two. It has not been in progress long enough to give the slope of the curve very accurately. As drawn in fig. 3 the growth in $\sqrt{8}$ years is 7·5 units, or 18·4 units per kilogram. The calculated value of λ_2 is $3·8 \times 10^{-5}$, and of $1/\lambda_2$ 26,000 years.

The experiment is of interest in this way. If the growth of radium had been proceeding at the same rate as in Experiment I., in $\sqrt{8}$ years 10·6 units of radium should have been produced, whereas the *total* amount of radium now present is less than this, being so far as the measurements indicate only 8·5 units. Even if it is assumed that this difference is due to errors of experiment, it is difficult to believe that the solution as initially prepared contained absolutely no radium. The results of the three experiments taken together thus indicate that the rate of growth of radium in terms of the square of the time is less for the first three years than for the subsequent year. This points to the existence of one or more intermediate bodies of relatively short period in the series, which would retard the rate of production of radium initially.

The actual measurements of the initial quantity of radium in Experiment I. bear this view out, although it must be remembered that the errors of measurement initially were very much greater than now, and this evidence thus only has a doubtful value. For Experiment II. the initial measurements may be rejected, as the quantity of radium was almost too small to be within the range of the older methods. In Experiment IV. no initial measurements were taken. In the older tests a brass electroscope had been used, and this was calibrated at every test with reference to a γ-ray radium standard. As was pointed out (II. p. 286) the measurements so corrected were not nearly so regular as the actual observations. In light of recent results on the uncertainty attaching to γ-ray measurements, it is best to reject these calibration tests altogether and to assume that the sensitiveness of the electroscope did not vary. Recalculating from the table (II. p. 282) gives, according to present data, the value 12 for the constant of the instrument. The mean of the first nine tests over the first 309 days given in the table (II. p. 286) recalculated gives the initial quantity of radium as 15·8 units. The initial quantity indicated by the curve on the assumption that the rate of production in terms of the square of the time has been constant from the start, is about 12·5 units. This evidence so far as it goes thus bears out the view that new short-lived intermediate bodies exist.

VI. *The Effects of Short-lived Intermediate Products.*

The general effects of short-lived intermediate bodies may now be considered. Let A, B, C, D, E ... and λ_1, λ_2, λ_3, λ_4, λ_5 ... denote the quantities and radioactive constants respectively of a parent element and its successive products. It is assumed that initially the products are absent, and that the change of the parent is so slow that its quantity may be considered constant. The differential equations connecting the terms are

$$\frac{d\mathrm{B}}{dt}=\lambda_1\mathrm{A}-\lambda_2\mathrm{B}\,; \quad \frac{d\mathrm{C}}{dt}=\lambda_2\mathrm{B}-\lambda_3\mathrm{C}\,;$$

$$\frac{d\mathrm{D}}{dt}=\lambda_3\mathrm{C}-[\lambda_4\mathrm{D}]; \quad \frac{d\mathrm{E}}{dt}=\lambda_4\mathrm{D}=[\lambda_5\mathrm{E}].$$

If we assume that the periods of the first two products, B and C, are short and those of the others long the terms in square brackets may be neglected. The case then corresponds to the disintegration series Uranium → Uranium X → "Uranium A" → Parent of Radium → Radium, where B denotes uranium X, for which λ_2 is about 11·4 $(\text{year})^{-1}$, C the suspected new intermediate body "uranium A," for which the period is probably of the order of unity, D the direct parent of radium, and E radium.

Solving the equations, the growth of radium is given by

$$E=\lambda_1\lambda_4 A\left\{\frac{t^2}{2}-\left(\frac{\lambda_2+\lambda_3}{\lambda_2\lambda_3}\right)t+\frac{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2}{\lambda_3^2\lambda_2^2}\right.$$

$$\left.+\left(\frac{\lambda_2}{(\lambda_3-\lambda_2)\lambda_3^2}\right)\epsilon^{-\lambda_3 t}-\left(\frac{\lambda_3}{(\lambda_3-\lambda_2)\lambda_2^2}\right)\epsilon^{-\lambda_2 t}\right\}. \quad \text{(I.)}$$

The last term is always very small, and may be neglected at once. The penultimate term approaches to zero as t increases. After a period several times that of uranium A it may be neglected. The expression then becomes

$$E=\lambda_1\lambda_4 A\left\{\frac{t^2}{2}-\left(\frac{\lambda_2+\lambda_3}{\lambda_2\lambda_3}\right)t+\frac{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2}{\lambda_3^2\lambda_2^2}\right\}.$$

If instead of reckoning the time from the start we reckon it from a date later by the sum of the periods of average life of the two short-lived bodies, that is, for t in the equation we substitute T, where

$$t=T+\frac{1}{\lambda_2}+\frac{1}{\lambda_3},$$

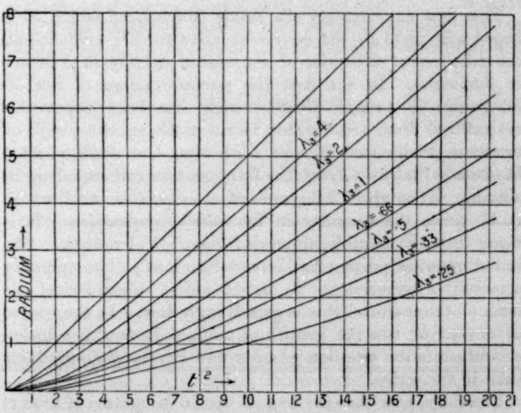
then

$$E=1/2\ \lambda_1\lambda_4 A\left\{T^2+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}\right\}=1/2\ \lambda_1\lambda_4 A T^2+\text{constant.}$$

That is to say, when the short-lived intermediate bodies come into equilibrium the production of radium proceeds strictly according to the square of the time reckoned, not from the start, but from a date later by the sum of the periods of the intermediate bodies.

To show the effect of the intermediate bodies initially the graphs of equation (I.) have been plotted against the square of the time, for the following values of λ_3 :—4, 2, 1, ·66, ·5, ·33, and ·25 (fig. 4). The curves therefore show the production of radium according as the hypothetical body uranium A has the period of three months, six months, 1, 1·5, 2, 3, and 4 years. The straight line uppermost on the diagram shows the production of radium on the same scale if no short-lived intermediate products intervened. It will be seen at once from these curves that if the portion between the third and fourth years, that is between $t^2=9$ and $t^2=16$, is examined, even for the lowest curve corresponding to the four-year period, the curve departs but little from the straight line. It is doubtful whether the departure would be experimentally detectable over this period. This

Fig. 4.



shows that the straightness of the experimental curve over the period available is no argument against the existence of intermediate bodies. On the other hand, the figure at once reveals the serious error introduced into the calculated period of the parent of radium by the formula used in Section V. if intermediate bodies exist. The true period of this body is obtained from the slope of the straight line uppermost in the diagram, which represents the rate of production of radium if no intermediate bodies intervened. If an intermediate body with a period of 1 year intervened, the period of the parent of radium instead of being 18,500 years as calculated in Section V. would really be only 0·72 times this or 13,300 years. If the period of the intermediate body were 4 years, the true period of the parent of radium would be only 0·32 of the apparent period, or 6000 years. Yet even a four-year period body might exist without producing in the curve a departure from the straight line enough to be detected experimentally with certainty as far as the measurements have yet gone.

VII. *New Experiments.*

Two new experiments are being commenced which it is hoped will yield in the course of time further evidence on the existence or otherwise of short-lived intermediate bodies in the series. In the first the purest fraction of the 50 kilograms of uranyl nitrate which has been repeatedly crystallized from fresh water in the work on the γ-rays of uranium (Soddy and Russell, Phil. Mag. Oct. 1909, p. 620) has been withdrawn from the fractionation and sealed up in a flask. It contains 3 kilograms of uranium (element), which is 12 times the quantity in the oldest preparation. It is being tested at monthly intervals for growth of radium. The initial quantity present has been determined in two agreeing consecutive experiments to be 42 units. Considering the mass of the material this is gratifyingly low. In the course of a year or two the growth of radium from this solution should settle the question whether new intermediate products exist in the series.

Secondly, now that the rest of the 50 kilograms of uranyl nitrate has been repeatedly purified, it is proposed to seal up one of the preparations of uranium X to be separated from it, as in the work with Mr. Russell, and to measure from it the rate of production of radium. Unless intermediate bodies exist this would hardly be worth while, as it would be practically impossible to distinguish the parent of radium formed from uranium X—if it is so formed—from that initially separated with the uranium X. Keetmann (Inaug.-Dissert. d. Verf., Berlin, 1909) has stated that uranium X and the parent of radium both have the same chemical properties as thorium and cannot be separated from this element or from one another. But if an intermediate body exists of period of the order of a year it should be easy to distinguish between the parent of radium initially present with the uranium X and that subsequently produced from it. If U represents the quantity of uranium initially in equilibrium with the uranium X separated from it, and R represents the quantity of radium formed from the uranium X, it can be shown that

$$\frac{R}{U}=\frac{\lambda_1\lambda_4}{\lambda_3-\lambda_2}\left\{t\left(1-\frac{\lambda_3}{\lambda_2}\right)+\frac{\lambda_3}{\lambda_2^2}(1-\epsilon^{-\lambda_2 t})-\frac{1}{\lambda_3}(1-\epsilon^{-\lambda_3 t})\right\},$$

assuming the same disintegration series as before. Whereas if all the parent of radium were present initially, the production of radium would of course proceed proportionally to the time.

VIII. *Conclusions.*

The measurements on the growth of radium in the three uranium solutions purified by Mr. T. D. Mackenzie between three and four years ago, have shown that in all of them the growth of radium is proceeding according to the square of the time within the error of measurement. The period of the direct parent of radium calculated from these results, on the assumption that no other intermediate bodies intervene, is 18,500 years in the case of the oldest solution for which the most complete data are available. But in the solution prepared last, the total quantity of radium now present is less than what would have been produced from the radium, assuming the rate of production to have been the same as in the first solution. This suggests the existence of at least one new product "uranium A" intermediate between uranium X and the parent of radium, with the period of the order of one year. From a mathematical investigation of the effect of such a body on the growth of radium it is concluded that it would not, if it existed, appreciably alter the production of radium according to the square of the time over the period observations have been made, but it would vitiate the calculations of the period of the average life of the parent of radium according to Rutherford's formula. In conclusion, it may be mentioned that a good deal of additional evidence bearing directly on this question of the existence of new intermediate products has been accumulated in an investigation on the rays and product of uranium X, with which it is convenient to deal in a separate communication.

Physical Chemistry Laboratory,
University of Glasgow,
October 1909.

ABSTRACT.

The measurements on the growth of radium in the three uranium solutions purified by Mr. T. D. Mackenzie between three and four years ago have shown that in all the growth of radium is proceeding at a rate proportional to the square of the time within the error of measurement. The methods of testing have been improved and the ordinary error is not greater than 10^{-12} gram of radium. This result indicates the existence of only one long-lived intermediate product in the series between uranium and radium. The period of average life of this body, calculated on the assumption that no other intermediate bodies exist, is 18,500 years in the case of the oldest solution for which data are available over a period from the end of the third to the end of the fourth year from purification. But for the solution prepared last, over a period from the end of the second to the end of the third year, the period indicated is about half again as long as in the first experiment. Indeed, had this solution grown radium at the same rate, with reference to the square of the time, as the older solution has been doing during the past year, more radium should have been formed than the *total* amount now actually present. This suggests the existence of at least one new intermediate product in the series "Uranium A" with a period comparable to the time observations have been in progress. From a mathematical investigation of the effect of such a body on the rate of growth of radium, it is concluded that it would not, if it existed, appreciably alter the production of radium according to the square of the time over the period accurate observations have been made, even were the period of the new body as great as four years. But its existence would vitiate the calculation of the period of the direct parent of radium according to the simple formula neglecting short-lived products. Other evidence on the problem is contained in the next paper (Rays and Product of Uranium X).

IV. *The Rays and Product of Uranium X.*

By FREDERICK SODDY, *M.A.*[*]

THE preceding paper has dealt with the evidence from the rate of production of radium from uranium, which suggests the possibility of the existence of at least one new intermediate product in the disintegration series. Although the experiments are not yet complete, it is advisable also to publish a short account of the results obtained in a different field of investigation which bear also upon the same problem. Attempts have been made to determine whether, during the decay of the β-radiation of the intensely active preparations of uranium X, separated with the help of Mr. Russell (Phil. Mag. Oct. 1909, p. 620) from 50 kilograms of pure uranyl nitrate, there occurred the growth of a feeble α-radiation increasing concomitantly as the other decayed. This is to be expected from the work of Boltwood, Keetmann and others, who have shown that the direct parent of radium gives an α-radiation, provided that it is the direct product of uranium X. The work was started in the hope that intermediate bodies intervened, and that the growth of α-rays would occur long enough after the β-rays of the uranium X had decayed to be detectable by ordinary measurements. But through the generosity of the friend who provided the uranium a powerful electromagnet of the Du Bois type was acquired, and with this it was attempted to deviate the β-rays of the uranium X, so that measurements of the α-rays of the preparations could be taken from the start. The problem proved a difficult one, but gradually the experimental methods have been perfected, so that now it is claimed that they are sufficiently delicate to detect such a growth of α-rays if it occurred in the intensely active uranium X preparations investigated to the extent to be expected from theory. Much remains unaccountable in connexion with these experiments, but they appear to have definitely answered the main question in the negative. No production of α-rays, such as is to be expected from theory, has occurred, and therefore it does not seem possible that the direct parent of radium can be the direct product of uranium X. The experiments indeed raise the further question whether the parent of radium can be a product of uranium X at all, but are not yet far enough advanced to be conclusive.

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[*] Preliminary accounts of part of this work appeared in two letters to 'Nature,' Jan. 28th, 1908, 79, p. 366, and March 11th, 1909, 80, p. 37. The paper was communicated to the Physical Society of London, on Oct. 22nd, 1909.

Were the parent of radium a product of uranium X the number of α-particles it emits should either be equal to, or one-half of, the number emitted by the uranium in equilibrium with it, according as one or two α-particles are given off from uranium. The latter assumption will be made as it gives the smaller growth of α-rays theoretically to be expected from uranium X. Assuming that the α-activity is proportional also to the ranges of the respective α-particles, which are 2·7 and 3·5 cm., the α-activity of the parent of radium will be $2{\cdot}7 \div 3{\cdot}5 \times 0{\cdot}5$, or ·38 of that of the uranium. The period of the parent of radium as deduced in the last paper on the assumption that no new intermediate bodies exist in the series, is 18,500 years, while that of uranium X is about 32 days, or 5×10^{-6} in terms of the other as unity. The amount of uranium X in equilibrium with a definite amount of uranium should produce therefore upon complete disintegration about 5×10^{-6} of the amount of the parent of radium in equilibrium with the same quantity of uranium. The α-activity of this product would therefore be about 2×10^{-6} of that of the uranium. In other words, the uranium X in equilibrium with 1 kilogram of uranium should thus produce in its complete disintegration a product having the α-activity of 2 milligrams of uranium.

The uranium X, after concentration to the greatest possible extent, was finally prepared in the form of films in rectangular trays having an area of 10 sq. cm., which could be slipped between the poles of the electromagnet. The undeviated radiation entered an electroscope placed above the poles provided with an opening in its base covered with very thin aluminium-foil. The first attempts were frustrated and the first preparations decayed without result, owing to the totally unexpected difficulty in deviating the β-rays.

The latter previously had been supposed to consist of homogeneous rays having a value for $H\rho = 2000$. But it was found that a field twice as strong as was necessary completely to deviate such rays failed to deviate about 5 per cent. of the total β-radiation. In the later dispositions the width of the gap was reduced to one-half the former width, and the field was sufficient to deviate completely all rays with a value for $H\rho$ below 8640 ; but the leak due to the still undeviated β-radiation, although now proportionately very small, was still several times that due to the γ-radiation. It was still too great in the earlier measurements for the small growth of α-radiation, theoretically to be expected, to be within the range of certain detection. In the later experiments the electroscope was filled with hydrogen instead of with air, and this constituted an enormous advance. Not only is the ionization due to the β- and γ-radiation greatly reduced and that due to the α-radiation greatly increased, but the convection-currents generated in the electroscope by the warming up of the magnet are also enormously reduced. The readings are far more regular and can be continued for a longer time before the leaf begins to be disturbed. In air, after half-an-hour's work or less, the leaf commences to wave like a flag, and nothing can then be done for eight hours. A preparation which gave (in divisions per minute) a leak of 370 divisions bare, and 351 covered with thin mica (α-rays therefore 19) when the electroscope was filled with air, gave in the same apparatus filled with hydrogen, bare 105·2 covered 66·2 (α-rays therefore 39). As the subsequent history of the preparation showed, a large proportion of the radiation reckoned as "α-" in air is due to the slight absorption of the β-rays in the mica, whereas in hydrogen this absorption is nearly negligible. So that in air, even if the theoretical growth of α-radiation occurred, it would probably be largely, if not entirely, masked by the diminution in the difference between the two leaks, covered and uncovered, due to the diminution in the absorption of the β-rays as these decayed. In hydrogen the effect is still present to a slight extent but not enough to prevent the observation of the growth of α-rays, if it occurred at the theoretical rate. Subsequent work has thrown doubt on the earlier measurements in air, and on the conclusion drawn from them in the second communication to 'Nature,' that in one preparation a rapid growth of α-rays occurred which reached a maximum in 2·5 days from the start. This observation has not been repeated either in air or in hydrogen with more perfect methods. This is not itself conclusive against it, for it has been impossible to prepare two sets of uranium X preparations in the same way. The chemical methods, of separation from uranium and of concentration from unidentified impurities, employed have varied widely, being dictated by circumstances from moment to moment during the process, owing to the perpetually changing character and amount of impurities and the totally different chemical behaviour of the uranium X accordingly. In fact the methods adopted in the last separation were often the converse of those formerly most relied upon, the uranium-free, uranium X-containing substances being *dissolved* in excess of ammonium carbonate and fractionally precipitated by boiling. Sometimes the uranium X came down in the *last* minute fraction almost entirely, and in other cases in one of the middle fractions. This new method proved an extremely valuable one in the last separation. Unfortunately it was not discovered before the whole of the uranium X had been first separated by the tedious barium sulphate process.

The earlier observations and the conclusion referred to may be rejected at least for the present. The difficultly deviable rays behave in an anomalous manner which was at first not suspected, in that they do not seem to be a constant quantity. In the case of the most recent preparation, which, as already explained, differed in the manner of its separation from the earlier ones, two successive measurements of the difficultly deviable radiation from the covered preparation in air, the first immediately after preparation, and the second 9 hours later, showed a diminution in the interval in the ratio of 4 to 3. The γ-rays of the preparation, measured in the usual way in a lead electroscope at the same times, showed no perceptible variation. It was later found that the value of the leak due to the difficultly deviable radiation varied in a completely unaccountable way. The variation is far more marked in air than in hydrogen. The behaviour suggested strongly in certain cases that the preparation was giving off a radioactive emanation, and from the manner of its preparation thorium was suspected. But this was completely disproved by special experiments. But the most striking proof that the behaviour was not in any way due to this case or to defects in the method of measurement is that the α-radiation as measured by the difference between the leaks with the preparation covered and uncovered remained from the start perfectly constant. This shows that all the numerous errors to be guarded against in experiments where the gas in the instrument is changed have been avoided.

A good many rapidly improvised experiments, done with the first batch of preparations before their activity decayed, had indicated that the difficultly deviable radiation was entirely similar in general character, both in the direction of their deviation and in the value of their absorption coefficient, to the ordinary β-rays. The absorption coefficient for aluminium, determined by placing strips of foil directly over the preparation, placed in the magnetic field, had been found to be about 24 $(\text{cm.})^{-1}$. This is much *greater* than the usual value (14), given for the ordinary β-rays, and is about equal to the value after 2·5 mm. of aluminium have been traversed. The field would cause the trajectories of the rays in the metal to be lengthened, but owing to the complete scattering suffered by these rays by even thin films of metal, it is difficult to state precisely what the effect of the field on the absorption coefficient would be. If these rays are really β-rays, the value of $H\rho$ shows that they must possess a velocity practically that of light, and a kinetic energy many times that of the ordinary β-rays. In view of the great decrease in penetrating power which a very slight diminution of velocity appears to produce in this type of radiation, it might be expected that the difficultly deviable rays would have a very great penetrating power, whereas the value found was actually less than the normal value. Similar difficultly deviable radiation is given out also by radium, although here the value of $H\rho$ is even higher, and the upper limit does not appear to be reached at 9000 to 11,000. Further study may show that they are to be connected with the scattering of the β-rays, though the experiments done rather point to the view that they may be a new kind of radiation. So far attention has been concentrated rather upon the main problem.

The electroscope used for hydrogen had a very thin mica screen, thick enough to absorb α-rays, hinged inside, which could be turned up or down from the outside, without opening the instrument, to uncover or cover the preparation at will. The first measurements, with the third set of preparations, were interfered with somewhat by difficulties encountered in making and keeping the electroscope perfectly gas-tight. Later a new instrument with properly soldered joints throughout was constructed and is now used entirely. There were three preparations in the third set, two practically identical in every particular, containing initially the uranium X in equilibrium with 5·1 kilograms of uranium, and a third containing that of 2 kilograms, as measured by means of the γ-radiation. They all showed the same behaviour, and one only of the two stronger preparations need be considered here. It weighed 200 milligrams. Measurements were done in hydrogen only. No evidence of a growth of α-rays either initially or at any subsequent time was obtained. The mean of the first five observations over the first 28 days gave 14 for the α-rays, which, corrected for the difference between the two instruments, gives 16 for the leak in terms of the present instrument. The actual observations varied a few per cent. from the mean for the reason stated. Subsequent measurements with the new electroscope 76, 99, and 137 days from the start, gave for the α-radiation 17, 17·2, and 16·6 respectively.

Time (days)	0	0·5	1·0	2·0	3·0	4·0	6	9	18	33
α-rays	38·6	39·7	38·8	38·9	39·0	39·2	37·2	37·1	39·8	40·5

In the fourth separation there was one preparation only, weighing 77·6 milligrams, and containing initially the uranium X in equilibrium with 5·05 kilograms of uranium. The measurements of the α-radiation were done at first both in hydrogen and air. Those in hydrogen, at different times from preparation, are given in the table on preceding page. They are the most accurate so far done and are all comparable with one another.

The initial value of the leak due to the penetrating rays was 75. It will be seen that from the start over a period of 33 days—about the period of average life of uranium X, in which 0·632 of the total disintegrates—the α-radiation has been practically constant. To obtain an idea of the growth of α-rays theoretically to be expected, the α-radiation of a film containing 25 mg. of uranyl nitrate (=12 mg. uranium) was measured in hydrogen under precisely similar conditions. It gave an α-ray leak of 17·8. The growth of α-rays to be expected in the last experiment in 33 days, if the the change of uranium X into the parent of radium occurred directly, is equal to that of $0{\cdot}632 \times 5{\cdot}05 \times 2 = 6{\cdot}5$ mg. of uranium, or about 9·6 divisions. Even if a considerable reduction is made for the fact that the weight of the uranium-X film was about three times the weight of the uranyl nitrate film, it is difficult to believe that the growth would not easily have been detected in the experiment if it had occurred.

In the preceding experiment with the preparations of the third separation, the initial α-activity was less than half the value in the last experiment. Any rise could therefore have been still more easily detected. But up to the period of 137 days from preparation, when less than 2 per cent. of the original uranium X remains undisintegrated, not the slightest growth has been recorded.

The most active preparation of the second separation tested recently in the new hydrogen electroscope in hydrogen on two occasions 200 days and 227 days from preparation, gave the same α-ray leak (about 80). The best preparation of the first separation—the best one ever prepared from the point of view of smallness of mass—tested in air (in a much weaker field than now used) has latterly, now that the β-radiation has nearly decayed, shown no change in the considerable α-radiation present. In the last tests 264 and 291 days from preparation, the α-leak has been constant at 35 (in air). This and a still earlier preparation now 16 months old, prepared from 3 kilograms of uranyl nitrate in preliminary work, have now decayed so far that their α-radiation can be examined without the aid of a magnet in an ordinary electroscope against suitable standards. Since these tests were started the α-radiation has remained constant. A still older preparation, dating from 1903/4, prepared from probably not more than 50 grams of uranyl nitrate, has also quite an appreciable α-radiation. This has been kept under careful observation for over a year and no change has been recorded.

These tests, therefore, although still incomplete, cover for different preparations the period from the start up to nearly a year in the case of the main preparations, and for considerably longer for weaker preparations. They have given no evidence of the growth of an α-radiation at any time during this period. They all show a considerable constant α-radiation, but in all it appears to be due to a body present from the start, unconnected genetically with uranium X. The statement of Keetmann, that uranium X and the parent of radium are identical in chemical behaviour and are always separated together, would explain the presence of the α-ray body in all the preparations so far kept under observation. The conclusion seems to be justified that the parent of radium is not the direct product of uranium X, and the question now arises whether it can be considered a product of uranium X at all. Even if there was a new intermediate product in the series "uranium A," with a period of many years, provided there was but one, some indication of growth of α-rays should before now have been obtained in these experiments. For it must be remembered that its existence would greatly reduce, as explained in the last paper, the period of the parent of radium, as calculated from the measurements of the rate of growth of radium, and so would increase correspondingly the growth of α-rays ultimately to be expected. The subsequent history of the preparations may be expected to throw further light on this question.

The results so far obtained are difficult to reconcile with the experiments on the rate of production of radium from uranium, on the assumption that the parent of radium is a product of uranium X. There is no proof of this, though on account of the intense character of the β-rays of uranium, it seems natural to suppose that uranium X, the β-ray-producing body, is in the main radium series rather than in that of actinium.

Physical Chemistry Laboratory,
University of Glasgow.
October 1909.

ABSTRACT.

Experiments have been made with the uranium X preparations separated with the help of Mr. A. S. Russell from 50 kilograms of pure uranyl nitrate (Phil. Mag., Oct. 1909, p. 620) to see whether there occurred the growth of a feeble α-radiation as the intense β-radiation decayed. Such a growth of α-rays, concomitant with the decay of β-rays, is to be expected if the parent of radium is the direct product of uranium X. From the period of the parent of radium given in the last paper, the uranium X in equilibrium with 1 kilogram of uranium should give by its complete disintegration a product having the α-activity of 2 milligrams of uranium, if no new intermediate bodies intervened.

The preparations of uranium X were examined in a magnetic field sufficient to deviate all rays having a value for $H\rho$ less than 8640, but the still undeviated β-radiation produced a leak in the electroscope several times greater than that due to the γ-rays. So far as can be seen, these difficultly deviable β-rays are similar in general character and in the value of their absorption coefficient, to ordinary β-rays. The first measurements were made in an electroscope filled with air, by covering the preparation with a thin screen, sufficient to absorb α-rays and measuring the difference between the leaks with and without the screen. The results of these experiments are believed now to be untrustworthy, and they are rejected provisionally. In later experiments the electroscope was filled with hydrogen, which constituted an enormous advance, and these experiments have shown that the α-radiation of the preparation remains sensibly constant as the β-radiation decays. Anomalies have been encountered with the difficultly deviable β-radiation, which appears to vary in intensity according to the conditions in an unexplained way. But throughout, the "difference leak" between the preparation bare and covered, due to α-rays, has remained constant in all the preparations examined. These measurements of the α-rays, for different preparations, cover a period from immediately after preparation to nearly a year in the case of the main preparation, and longer periods in the case of weaker preparations. The two most recent preparations each contained the uranium X in equilibrium with about 5 kilograms of uranium, and the growth of α-rays if the change of uranium X into the parent of radium were direct, should be equal to the α-radiation of 10 milligrams of uranium.

This should have been easily detectable under the conditions of the experiment. It is concluded that the parent of radium cannot be the direct product of uranium X. The experiments, taken in conjunction with those given in the preceding paper, indicate that it is not a product of uranium X at all, but the subsequent history of the uranium X preparations must be awaited before this can be decided.

DISCUSSION.

Prof. STRUTT remarked that the Author's experiments covered much ground, and suggested many interesting questions. He congratulated Mr. Soddy upon the skill with which he had attacked the subject, and referred to the difficulty of carrying out successfully complicated experiments which extended over long periods of time. With reference to the experiments on the growth of radium, he asked the Author what was the ratio between the amount grown and the actual quantity present at the start, and also what multiple was the final leak of the electroscope of the normal air-leak. With regard to detecting the growth of α-radiation in the presence of β-radiation, he suggested that the scintillations produced by α-rays on a phosphorescent screen might be made use of. Referring to the experiment on the growth of helium in sylvine, Prof. Strutt remarked that he would have been astonished if the Author had obtained any evidence of the production of helium in the comparatively short time over which his experiments extended.

The CHAIRMAN drew attention to one of the formulæ used by the Author in which the "time" occurred as the "square." He asked if this formula held good when the time considered was large.

Mr. SODDY, in reply, said the natural leak of the electroscope was 1·1 (divisions per minute), and the leak due to the radium in the oldest preparation at the end of the fourth year was 4·2. The initial value found by extrapolation from the curve shown, in which quantity of radium was plotted against the square of the time, was 2·1. The initial value found experimentally as the mean of the first nine observations extending over the first nine months, when the methods were less sensitive than now, was 2·7. This bears out the view that a new intermediate body exists, retarding the initial rate of production of radium, though no great weight can be attached to the initial observations.

Professor Strutt's suggestion that the α-radiation of uranium X preparations should be examined by the scintillation method might prove a valuable one, though on account of the impossibility of entirely removing β-rays it had not so far been attempted.

With reference to sylvine, the crystalline character of the mineral rendered it open to question whether helium if generated would be retained. In addition the radioactivity of potassium had suggested the experiment. No doubt a production of helium sufficient to detect would be remarkable, but it seemed worth looking for.

Answering the Chairman's question, the theoretical law, that the production of radium should proceed at a rate proportional to the square of the time, only referred to the initial rate of production, that is for periods of observation short in comparison with the period of average life of the direct parent of radium. The latter must be many thousands of years, and the law referred to should hold accurately over a far longer period than could be examined in the life-time of a single observer.

V. *Note on a Gravitational Problem.*

By C. V. Burton, *D.Sc.*[*]

1. Without actual loss of generality, the question here considered may be stated as follows :—*If the action of gravity were intermittent in character, the mutual attraction of any two bodies fluctuating between zero and twice its mean value, how long could the period of the fluctuation be without giving rise to observable periodic effects?* At a first glance we might be tempted to reply that the period could only be a very small fraction of a second ; but though this answer would be justified if the earth were perfectly rigid, the yielding of the earth to pressure and to shearing-stress completely changes the aspect of the problem, and the results obtained are so different from what might have been hastily assumed, that a brief indication of them may possess some interest.

2. Let us suppose that the gravitation-constant G represents merely the mean value of $G(1+Ae^{ipt})$, where A is a numerical constant ; so that, at any time t, the attraction between masses m, m' at a distance r apart is

$$G(1+Ae^{ipt})\,mm'/r^2.$$

For the sake of simplicity the earth will now be regarded as spherical and sensibly homogeneous ; and accordingly the intensity of the gravitative field at any point of the surface at time t will be

$$g(1+Ae^{ipt})\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and at any interior point distant r from the centre

$$\frac{gr}{a}(1+Ae^{ipt}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

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[*] Read October 22, 1909.

3. At any interior point, a mass-element *free* to move under the periodic term of the field-intensity (2) would execute oscillations given by

$$\delta r = -\frac{gr}{ap^2} Ae^{ipt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Consider then the radial vibrations defined by (3), and the forces which would be needed to maintain them. At any given instant, every radius vector r has acquired an increment δr proportional to r, the whole sphere being thus symmetrically and homogeneously expanded or contracted, and the proportional increment of linear dimensions being $-g/ap^2 . Ae^{ipt}$, corresponding to the proportional increment of volume

$$-3g/ap^2 . Ae^{ipt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the sequel the discussion is restricted to the case where Ap^2 is small enough to make the expression (4) a small fraction.

4. Thus if every mass-element suffers the periodic changes of position defined by (3), there will also be for each element periodic (proportional) changes of volume indicated by (4). The changes of position we may consider to be provided for by the periodic term in (2), while throughout the mass of the planet, the fluctuations of pressure needed to produce the changes of volume of the various elements are supplied by the reaction of those elements on one another. At the free surface, however, we should further require a periodic pressure P given by

$$P = 3kg/ap^2 . Ae^{ipt} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

to complete the system of forces under which the vibrations (3) would be executed; k being the bulk-modulus of elasticity of the substance of the planet.

5. For a moment suppose the periodic pressure P to be applied to the earth's surface, in addition to the partly periodic character of gravitation expressed by the factor $1 + Ae^{ipt}$. Then every mass-element of the earth would be executing a vibration expressed by (3), precisely as if it were free to move under the periodic part of the earth's attraction, and a relatively small body resting on the earth's surface would

execute, *under gravitative influence alone*, vibrations exactly agreeing with those of the surface. There would accordingly be no periodic term in the reaction between the body in question and the ground upon which it rested; the weight of the body would appear perfectly constant, and, generally, no observations concerned wholly with gravitating bodies at the earth's surface * would give any indication of periodic variations in the gravitation "constant."

6. Now further add to the system of forces acting on the earth a superficial periodic pressure $-\mathrm{P}$; thus the sole periodic influence at work is that due to the term $\mathrm{GA}e^{ipt}$ in the gravitation "constant," while the only effects observable at the earth's surface are such as arise from the pressure $-\mathrm{P}$. The result of such a periodic pressure is a radial movement of the surface of period $2\pi/p$, and of an amplitude which, under certain simplifying assumptions, can be readily expressed in terms of the elastic constants of the earth's substance.

7. Homogeneousness has already been assumed; let us now further postulate some small degree of viscosity, sufficient to ensure that compressional waves of short period (one second or less) propagated inward from the surface, would be sensibly extinguished before reaching the centre, though not sufficient to produce any perceptible damping effect in the course of a few wave-lengths. We have thus only to consider the waves travelling *inward* from the surface under the influence of the periodic pressure $-\mathrm{P}$; and since, in the cases to which this inquiry is restricted, the wave-length of the disturbance is very small compared with the earth's radius, the curvature of the surface may be disregarded, and the problem of the motion near the surface reduces to one of plane waves.

8. Let $a-r$, the depth of any point below the surface, be denoted by h; the waves due to the pressure $-\mathrm{P}$ and propagated in the direction of h-increasing may be represented by

$$\delta h = \mathrm{C}e^{ipt(t-h/\mathrm{V})}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the constant C has to be found and is not necessarily real; while the velocity V is given by

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* The attractions of such bodies for one another are here excluded from consideration.

$$V^2 = (k+\tfrac{4}{3}\mu)/\rho \; ; \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here k, as already defined, is the bulk-modulus of elasticity of the earth's substance, while μ is its rigidity, and ρ its density. The radial stress at any point, reckoned as a *pressure*, is

$$-(k+\tfrac{4}{3}\mu)\, d^2\delta h/dh^2 \; ;$$

and, equating the value of this for $h=0$ to $-P$ given by (5), we readily find

$$C = \frac{3ikgA}{8\pi^3 a n^3 \sqrt{\{(k+\frac{4}{3}\mu)\rho\}}}, \quad . \quad . \quad . \quad . \quad (8)$$

where $n=p/2\pi$, the frequency of the periodic variation of gravity.

9. The rigidity μ of the earth as a whole is not very different from that of steel*, and if we assume the compressibility also to be something like that of steel, we may take as roughly approximate numerical values

$$k = 1{\cdot}5 \times 10^{12} \text{ dynes/sq. cm.}, \quad \mu = {\cdot}82 \times 10^{12} \text{ dynes/sq. cm.} \; ;$$

moreover, ρ the density is about 6·6 grams per cc., and thus (8) gives for the semi-amplitude of the *apparent* motion of the earth's surface in centimetres

$$\cdot 0068 A/n^3.$$

10. As an example, take A equal to unity, and n equal to one second. The gravitative attraction between any two bodies would then vary between zero and twice its mean value, the period of the variation being one second. The *actual* periodic displacement of the earth's surface would amount to about 25 cm. above and below its mean position; but the *observable* effects would, with our assumptions, be the same as if (with gravity invariable) the earth's surface were pulsating with a period of one second and with a semi-amplitude of ·0068 cm. The maximum acceleration involved in this apparent pulsatory movement would be about ·35 cm./sec², or roughly 1/2500 part of the mean acceleration of gravity; and refined observations would be required to discern any periodic effects. If the frequency n of the

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* A somewhat vague statement, though definite enough for the purpose in view.

fluctuations were changed in a given ratio m (their amplitude remaining the same), the amplitude of the *apparent* surface pulsation would be changed in the ratio m^{-3}, and the maximum acceleration involved in that pulsation would be changed in the ratio m^{-1} *.

11. It appears, then, that the interaction of the earth with bodies at its surface is far from providing a delicate test for the existence of rapid periodic fluctuations of gravity. The simplifying assumptions which have been made above are not such as to alter the essential aspect of the problem ; no doubt the substitution of more correct (mean) values for the elastic constants involved would lead to somewhat different results, but the general character of the conclusions would remain as before.

12. The apparatus used in the Cavendish experiment, as carried out, for instance, by Mr. Boys, would indicate rapid fluctuations of gravity with far greater sensitiveness ; on the other hand, Prof. Poynting's form of the experiment, wherein a beam-balance was used, would be practically unaffected by such a state of things.

Kennington, near Oxford,
June 26, 1909.

Postscript: *Nov*. 17, 1909.

I have been reminded by Dr. Chree, President of the Physical Society, that my simplifying assumption of an elastically homogeneous earth must necessarily be far from the truth, the pressure at the centre being something like 2800 tons per square inch. This I am most ready to admit ; but although the assumption requires modification, we do not know to what extent. Tidal phenomena afford some information as to the rigidity of the earth as a whole, but it seems very probable, as Dr. Chree points out, that the bulk-modulus of elasticity of the more central portions is increased by the pressure thereon to a far greater extent than is the rigidity-modulus. This would reduce the amplitude of the *actual* compressional waves discussed in the paper, the *observable* effects at the earth's surface being thus correspondingly increased. On the other hand, it appears that my assumed values for k and μ give for Young's modulus a value considerably higher than is sometimes assumed, and this would affect the conclusions in the contrary sense.

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* The amplitude and extreme acceleration of the apparent motion are of course proportional to A, which, according to any pulsatory theory of gravitation, would be a large number. The value unity has here been given to A merely by way of illustration.

The problem was one which presented itself in the course of some speculations regarding the mechanism of gravitation; its solution was not afterwards found to be important in that connexion, but it seemed that, as a mere dynamical curiosity, the matter might be worth a short note. So in the above discussion the assumptions made were the simplest possible, and instead of the most recent authorities being consulted for geophysical data, Lord Kelvin's original rough comparison between the earth and a sphere of steel was made to serve, For the present it hardly seems that labour would be profitably expended on a more elaborate treatment.

VI. *The Absorption Spectrum of Potassium Vapour.*

By P. V. Bevan, *M.A., Royal Holloway College* [*].

[Plate II.]

A CERTAIN amount of work [†] has been done on the absorption of light by potassium vapour, but only in the visible region of the spectrum. The absorption in the ultra-violet as well as in the visible part is specially interesting, firstly for its similarity to the corresponding absorption by sodium vapour and secondly in relation to the analysis of lines into series by Kayser and Runge, and by Rydberg. The recent work by Wood [‡] has shown how the principal series lines for sodium appear in the absorption spectrum in much greater numbers than in any kind of emission spectrum that has as yet been studied. A similar effect takes place with potassium, and the author of the present paper has measured the wave-lengths of lines of the principal series up to the line corresponding to $n=26$ in the Kayser and Runge formula. The method of experiment was the very simple one of passing light through a steel tube containing potassium, the ends of the tube being closed by quartz plates, the emergent light was then examined with a quartz spectrograph. The tube could be heated and so a certain portion of it filled with potassium vapour of density varying with the temperature. Potassium under these conditions behaves in a very similar manner to sodium, and there is a field for work of a similar character to that of Wood on the optical properties of sodium vapour. The spectrograph used was by Hilger giving a spectrum of about 8 inches long from w.l. 2100 to the red end of the visible spectrum. For a source of light a Nernst lamp and an arc lamp were used. The Nernst lamp has advantages in giving a continuous spectrum, but when measurements of wave-lengths are to be made the arc lamp is most suitable, as in addition to giving more intense ultra-violet light it also provides a set of standard lines from which wave-lengths may be calculated.

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[*] Read November 12, 1909.

[†] Roscoe and Schuster, Proc. Roy. Soc. xxii. p. 362; Liveing and Dewar, Proc. Roy. Soc. xxvii. p. 132, etc., & xxviii. p. 352, etc.

[‡] Phil. Mag., Oct. 1909.

First of all with regard to the visible spectrum. When the tube is heated absorption is first noticed at the pair of lines w.l.'s 7699, 7655, and this is soon followed by channelled space absorption in the region 6000–6800. This part of the spectrum will be discussed later. The violet pair w.l.'s 4047, 4044 next appear and later another channelled space in the blue-violet region. Finally another channelled space in the orange-green appears. The absorption in the region of the lines 7699, 7655 increases, and the lines broaden, soon making one dark band which spreads until it joins the red channelled space which has in the meantime become a dark band, and finally all light from the red end of the spectrum to about w.l. 6000 is cut off. The blue-violet channelled space also soon becomes a dark band, and this joins the band formed by the widening of the lines 4047, 4044. Figure 1 shows a series of photographs taken with increasing density of vapour. The lines 7699, 7655 have disappeared, the whole space up to about w.l. 6000 being dark. In (1) the blue-violet channelled space is just appearing. The principal series lines are very definite and can be counted up to about $n=16$

($n=4$ for the line 4047). As we go down this series of photographs we notice how the blue-violet channelled space increases—the channellings do not appear well in the plate though they are quite evident in the negative. The red absorption, however, changes its limit on the yellow side very little. The absorption lines also broaden but the broadening is not symmetrical, the line being more diffuse on the less refrangible side. New absorption lines appear on each side of the main lines, and these seem to form parts of channelled space spectra which appear between the primary lines. In (5) the blue-violet channelled space has extended to meet the broadened 4047 line, and all the light is cut off in this region. The orange-green absorption extends on the more refrangible side but undergoes little change on the other. The sodium lines D 3303 and 2852 appear faintly owing to a certain amount of sodium present in the potassium as impurity.

The Principal Series Lines.

These lines are the most definite feature of the absorption spectrum, and as has been mentioned above they have been counted up to the line corresponding to $n=26$. As $n=3$ for the lines 7699, 7665, the author has been able to observe 24 of this series. Hitherto the members up to $n=11$ have been observed, so that 15 new lines have been added to this list. The wave-lengths are as follows:—

n.	Wave-length.	
10	2963·4	(2963·36 Kayser)
11	2942·8	(2942·8 „)
12	2928·0	
13	2916·6	
14	2907·6	
15	2900·4	
16	2894·6	
17	2889·7	
18	2885·9	
19	2882·9	
20	2880·3	
21	2877·9	
22	2875·8	
23	2874·1	
24	2872·5	
25	2871·1	
26	2870·0	

The three formulæ given by Kayser * give for the value of the wave-lengths for $n=20$

2876·2 ; 2876·4 ; 2879·4 ;

the departures from the observed values in the cases of $n=10$ are, however,

+1·05 ; −0·5 ; +0·76 ;

so that the agreement is as close as could be expected, and a recalculation of the constants is necessary.

The values for the wave-lengths are estimated to be correct to within ·2 of an Ångström unit. With a larger spectrograph giving more dispersion and a wider aperture, there is no doubt that many more lines of this series could be photographed.

There appears to be no trace of lines belonging to the associated series for potassium in any photographs which the author has taken. This fact is of importance as it shows that there must be a very great difference in the mechanism corresponding to the principal series and the others, and that sodium and potassium are exactly analogous in this way. Another interesting point in connexion with these lines is the fact that though a large number of lines of wave-lengths less than 3000 have been observed in the spark spectrum †, none of these lines after 2992·3 corresponding to $n=9$ appear.

The Red Channelled Space Absorption Spectrum.

The interest of this spectrum and the other channelled spectra connected with the absorption by potassium vapour lies in its similarity to the sodium absorption spectrum in the neighbourhood of w.l. 5000. This region seems unconnected with any bright lines in the various emission spectra and also unconnected with any of the series lines of Kayser and Runge, or Rydberg. There is, however, a certain amount of evidence to connect this and the corresponding spectrum of sodium with the principal series of Kayser and Runge. Wood has shown that there is an intimate physical relation between the mechanisms which produce the D lines and this channelled absorption, and it will appear that the potassium channellings are related to the first members of the principal series in a manner similar to the corresponding sodium spectra. With sufficient resolving power the channelled spectrum is seen to be made up of a large number of fine lines; the edges of what appear as bands are regions of greater absorption and lie towards the violet end of the spectrum. This spectrum was observed and the edges of the bands mapped by Roscoe and Schuster *. Fresh measurements have been made for comparison with the corresponding sodium spectrum. Figure 2 shows the general appearance of the spectrum under as high dispersion as the author was able to obtain. It is seen that the spectrum consists of a number of fine lines which towards the violet end of the spectrum are of equal intensity and arranged at approximately equal intervals—the fluted appearance is not observed here. Going from right to left we come to the places where the changes in intensity give the fluted appearance with definite edges, and as we come to the red end we have a succession of bands equally diffuse on both sides. On this spectrum the neon spectrum was photographed for purposes of measurement. No definite relations amongst the lines have been detected, but remembering the extreme complexity of the similar sodium spectrum there seems no chance of doing this until the fluorescent spectrum under various conditions of excitation has been studied, as has been done in the case of sodium by Wood. The definite edges of the bands have been measured and it is of interest to compare them with the corresponding edges in the case of sodium.

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* *Handbuch der Spectroscopie*, ii. p. 524.

† Eder & Valenta, *Denkschr. Wien*, lxi. 1894.

The edges of the bands are at wave-lengths given in the following table. The first column gives the number of the band as designated in fig. 2.

No.	Wave-length.
1	6412
2	6442
3	6473
4	6512
5	6543
6	6581
7	6621

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* *Loc. cit.*

These are what appear on the negative as quite definite bands. There are outside these limits places which might be edges but which are not definite enough and so have not been included in this table. The measurements are accurate within two or three units, and we may note in passing that differences between neighbouring edges are approximately 30 and 40 units. Doubtful edges at 6383, 6353, give the same differences, but no stress is laid on this point at present.

In the case of sodium we have for the wave-lengths of edges of the bands :—

4783
4810
4819
4837
4866
4895
4932
4962
4979
5002
5041
5048
5080
5087
5119
5126
5134

These numbers are taken from Wood's paper in the Phil. Mag. xii. 1906, p. 499.

If now we tabulate the ratios of the wave-lengths of these edges to the first member of the principal series for the two cases, *i. e.* for sodium D 5896, for potassium 7699, we obtain the following numbers :—

Potassium.	Sodium.
	·8112
	·8158
	·8174
	·8204
	·8252
·8328	·8301
·8367	·8366
·8408	8417
·8460	·8445
·8499	·8483
·8548	·8549
·8602	·8562
	·8615
	·8628
	·8682
	·8694
	·8707

All the edges of the bands for sodium, that appear in the photograph accompanying the paper of Wood's to which reference has already been made, are included in the above table. It is seen at once that there is a definite correspondence between the two cases. The actual closeness of pairs of numbers in the two cases may not indicate anything, but the fact that all the numbers for potassium fall in the region occupied by the numbers for sodium is of interest, and also the further fact that the spacing of the numbers is also roughly the same. This seems clear evidence that the mechanism at the back of the two cases is similar. When one remembers the changes in this type of absorption spectrum produced by circumstances apparently trifling it seems exceedingly probable that on fuller investigation the correspondencies between the two spectra may be much more definite than is at present observed. Much more detailed observation is necessary in the case of potassium, and it is to be desired that someone with the instruments necessary may be able to make these observations. It seems clear, however, that the two absorption spectra are analogous and that they are connected with the first members of the principal series.

Anomalous Dispersion by Potassium Vapour.

The author has shown [*] that anomalous dispersion takes place in the region of the first members of the principal series in lithium and potassium, and further experiments have been made with potassium to investigate other lines of the same series. Anomalous dispersion has been detected at the first four lines of the principal series for potassium, but it is very small in amount after the second line 4047. The effect can be easily detected on the negative, but as it is very small excepting at the first two members of the principal series it does not show sufficiently for a reproduction. The source of light was a fine slit illuminated by an arc light. One difficulty in the investigation is concerned with focussing the image of the horizontal slit on the vertical slit of the spectrograph. The focus cannot be good for more than one line at a time as a quartz lens must be used, but further experiments are to be made for this effect.

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[*] Phil. Mag., Sept. 1909.

I desire to express my thanks to the Government Grant Committee of the Royal Society for a grant which enabled me to obtain the quartz spectrograph used for the work of this paper.

ABSTRACT.

The method of studying the absorption spectrum was that used first by Roscoe and Schuster, and of late years elaborated by Prof. R. W. Wood. The fact that the optically dense vapour has really very small density makes it possible to heat the metal in a tube, and to have enough vapour to show strong absorption of light with very little actual distillation to the colder parts of the tube. A tube with quartz plate ends can thus be used and the absorption spectrum studied with a quartz spectrograph. The most evident feature of the spectrum thus obtained is the appearance of the lines of the Principal Series. None of the lines of the two associated series appear, but additional channelled space spectra which are unrepresented in the emission spectra. Fifteen new lines have been obtained in the principal series by this method.

In the visible region there appears a channelled space spectrum in the red. This shows definite edges of bands towards the violet end of the spectrum. The wave-lengths of the edges of these bands were measured. When the ratios of these wave-lengths to that of the first member of the principal series are found, the values are found to be in the same range as the corresponding ratios as deduced from Wood's measurements on Sodium absorption. This the Author regards as evidence of connexion between the channelled space spectrum and the principal series of lines. Anomalous dispersion takes place at the first four members of the principal series, but the amount of the effect rapidly diminishes with the order of the line. The remarkable feature of this absorption spectrum and also that of Sodium as examined by Wood, is the enormous difference in the properties of the principal series lines from those of other lines in the emission spectrum. Some of these other lines are in emission spectra far stronger than the higher members of the principal series, yet they do not appear at all in the absorption spectrum. This may indicate that the metallic vapour at the comparatively low temperature of these experiments is in quite a different molecular state from its state in a spark or flame, or that in these latter cases chemical action is going on and the emission spectrum is not a simple elementary spectrum.

DISCUSSION.

Dr. J. A. HARKER remarked that in studying the absorption spectrum of a substance by heating it in a tube closed at the ends, there is a region of fog at the ends of the vapour for which allowance should be made. You do not get the absorption spectrum of the vapour because the result is affected by the scattering produced by the particles in the fog.

Prof. C. H. **Lees** asked if the Author could state the nature of the absorption in the second and third series.

Mr. **Rayner** asked if there was any difficulty in getting the metal pure. He also asked if the Author could state approximately the temperature of the vapour, and whether there was any gas present in the tube.

Prof. **Bevan** in reply stated that he had observed no evidence of scattering, such as would be produced by the existence of fog in the tube. The amount of vapour used was very small, and therefore there could be little cloud effect. The temperature of the vapour was much below the boiling-point of potassium. With regard to the gas in the tube, he had tried hydrogen at atmospheric pressure and at low pressure and he had also used a vacuum. With regard to the purity of the metal the sodium lines did appear but they were easily recognized.

VII. *Some further Notes on the Physiological Principles underlying the Flicker Photometer.* By J. S. Dow, *B.Sc.*[*]

In a paper read before the Physical Society of London in 1906 (see Phil. Mag. August 1906; Proc. Phys. Soc. vol. xx.), the author discussed some of the phenomena of colour-vision which affect heterochromatic photometry, and referred to the theory, due to von Kries, of the action of the minute light-percipient organs on the retina known as the "rods" and "cones," by the aid of which some workers in photometry have endeavoured to explain the "yellow spot" and Purkinje effects.

It was pointed out that, when the comparison of two sources of light differing in colour was attempted, the reading of a photometer of the "equality of brightness" pattern would be found to depend upon:—

(*a*) The obliquity at which rays from the illuminated surfaces strike the eye.

(*b*) The distance away of the eye from the surfaces.

(*c*) The size of the surfaces.

It was, however, also found that, for some reason, the effect was much less readily perceptible in the case of a flicker photometer; and this suggested that the physiological basis of such instruments might be found to differ from that of photometers of the ordinary variety. In this communication the author wishes to summarise the results of some further experiments on this point which seem to throw some light on the theory of the flicker photometer: this, however, as Mr. A. P. Trotter has justly remarked, is essentially physiological, and is therefore correspondingly difficult to investigate.

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[*] Read November 12, 1909.

Before proceeding further, however, it may be well to recapitulate briefly the main points in this theory of the action of rods and cones on the retina, as it will be frequently referred to in what follows.

According to this theory, there exist on the retina two distinct varieties of minute light-perceiving organs, known, from their appearance, as the "rods" and the "cones" respectively. The rods, it is thought, are sensitive to light but unable to perceive colour, as such; they are, however, most sensitive to bluish-green light, probably in the neighbourhood of 0·51 to 0·52 μ. Light of all wave-lengths seems to be usually perceived by these organs as white *. These organs are also sensitive to very weak light but, as the illumination is increased, they become as it were saturated and do not respond further to increased stimulus.

The cones, on the other hand, perceive colour but are most sensitive to yellow-green light, probably usually near 0·58 μ, and, while they are insensitive to the very weak order of illumination to which the rods respond, they continue to become more active, under the influence of increased stimulus, once they have started, long after the rods have ceased to do so.

As explained in the paper referred to above, this theory seems to account very satisfactorily for the Purkinje effect. Moreover, an experiment described therein also suggested that, as the Purkinje effect only became very noticeable to the author, for small retinal areas, at illuminations below about 0·2 lux, this might be the order of illumination at which the cones finally go out of action, and cone-vision

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* According to a recent paper by Professor G. J. Burch (Proc. Roy. Soc. 1905) the retina, which, as a rule, seems to lack the power of colour-perception at very low illuminations, regains it if kept in the dark for some hours.

is more or less completely replaced by rod-vision. In view of what follows this figure is rather important, and we may next briefly refer in passing to a confirmatory experiment which acts as a useful check upon it.

According to the rod and cone theory, the yellow-spot effect is to be ascribed to the peculiar distribution of rods and cones over the retina. At the centre of the retina only cones appear to exist; portions of the retina more remote from the fovea are covered by a mixture of rods and cones, while the extreme peripheral portions of the retina contain rods only. It will be seen therefore, that when the brightness of two surfaces, illuminated by the aid of red and green light respectively, is compared, the result will depend very largely on the proportions of rods and cones on the portion of the retina on which the image of them is received. It therefore occurred to the author that it would be of interest to obtain curves connecting the distance away of the eye from the photometer-surfaces so illuminated, and the apparent relative brightness of the red and green, with a series of different illuminations of the photometer. This experiment was carried out in exactly the same manner as those of a similar description detailed in the paper before the Physical Society in 1906.

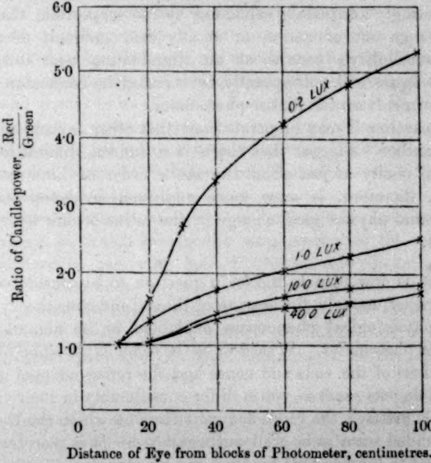
The results are shown in fig. 1, and seem to bear out completely the suggestion that, whatever be its explanation, a somewhat abrupt alteration in the nature of colour-vision occurs, in the case of the author's eye, near 0·2 lux.

The instrument was in this case a Joly photometer, the blocks of which were illuminated by red and green light furnished by two carbon-filament glow-lamps screened with ruby-red and signal-green glasses respectively. It will be seen that, as the eye is withdrawn, the reading invariably tends to favour the red light at the expense of the green. This, according to the rod and cone theory, is to be explained on the supposition that, as the image falls more and more towards the centre of the retina, the number of rods, as compared with cones, in the area covered by it, continually decreases; eventually, when the fovea centralis is reached, a more or less steady state of things should be arrived at since in this region of the retina only cones are believed to exist.

For convenience in comparing the steepness of the curves at different illuminations, all the readings are reduced to a scale such that the value corresponding with the eye at a distance of 10 centimetres is unity. It will be observed that, as we should expect, we obtain in each case a curve which gradually rises but afterwards tends to become horizontal.

Fig. 1.

Showing Effect of varying distance of Eye from Joly Photometer at various illuminations of the photometer-blocks. ("Equality of Brightness" Method.)



But it will once more be seen how very much steeper the curve suddenly becomes when the illumination is reduced to the order of 0·2 lux (metre-candles). On the other hand, an increase in the intensity of illumination from 10 to 40 lux has relatively little effect. All this is exactly what the theory would lead us to expect. It is apparently only at comparatively low illuminations that the struggle of the rods and cones becomes marked. And at an order of illumination about 0·2 lux (metre-candles) the cones might be supposed abruptly to cease exercising their powers, with the result that the relatively small number of rods in play become proportionally more effective. A small change in the retinal conditions would therefore be much more influential in disturbing the judgment of balance than at high illuminations, and the curve becomes correspondingly steeper.

The above experiment, therefore, is interesting not only because of the confirmation it seems to afford to the provisional rod and cone theory, but also because it serves to bear out the suggestion that an abrupt change in the retinal conditions (seemingly adequately explained by the supposition that the cones go out of action or at any rate suddenly decrease in sensibility) occurs at an illumination near 0·2 lux. This figure will subsequently be recalled in connexion with experiments on the flicker photometer.

Meantime it may be pointed out that other experiments of the author * suggest that there is a sudden diminution in visual acuity at just about the above order of illumination. This, therefore, is once more additional evidence that a profound physiological change in the retina occurs near this point.

Let us now turn to the main question to be considered in this paper, namely, the bearing of the rod and cone theory and the physiological phenomena explained by its aid, on the flicker photometer. Whatever be the exact explanation of the functions of the rods and cones and the retina adopted (and physiologists seem as yet to differ considerably in their views on this matter), the physiological effects on which the theory is founded seem to be well authenticated. It is therefore of interest to inquire how far they can be considered applicable to the flicker photometer, which is now beginning to receive a considerable amount of attention as a means of comparing sources of light which differ in colour. As an illustration of the possibility of such instruments yielding results which differ from those obtained by the aid of photometers of the

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* 'Illuminating Engineer' (London), vol. ii. April 1909, p. 237.

"Equality of Brightness" or "Contrast" types, an experience recently related by Mr. L. Wild may be mentioned. This observer finds that, when an incandescent lamp, having a tungsten filament and run at a consumption of 1·5 watts per C.P., is compared with a carbon-filament lamp running at 4 watts per candle, the result varies by 6% according as a flicker photometer or an instrument of the ordinary variety is used for the test. Apparently the tungsten lamp has a lower candle-power in the former case *.

This matter was touched upon in the writer's previous communication to the Physical Society on Colour-photometry †. On that occasion it was found that the readings of an Everett-Edgcumbe flicker instrument, when used to compare red and green lights, did not differ, when the distance of the eye from the field of view of the instrument was altered, nearly so greatly as was found to be the case for an "Equality of Brightness" instrument. This suggested, therefore, that the behaviour of the retina was in some way different, when judging disappearance of flicker, from that characteristic of it when judging *brightness*.

It seemed, however, preferable to confirm this result by an experiment in which exactly the same portion of the retina was utilized in observing both brightness and flicker, and the illumination of the surfaces viewed was the same in each case. Under these circumstances, the two sensations would be compared under exactly similar conditions, and if the physiological principle of both instruments is the same, should lead to similar results.

Comparison of the Yellow-spot Effect for Equality of Brightness and Flicker Photometers.

The Rood type of photometer is particularly convenient for the investigation of this point for, by keeping the lens still, one may use it as an ordinary instrument, and then, by oscillating the lens, compare the brightness of the two photometrical surfaces, still at the same distance from the eye, by a flicker method.

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* Electrician, July 16, 1909.

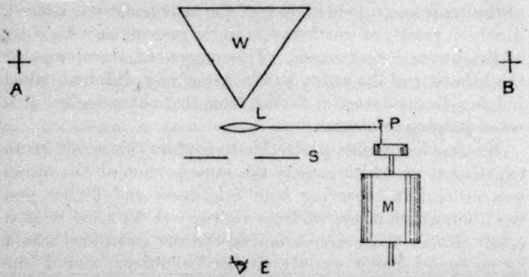
† Phil. Mag. vol. xx. 1906.

For this purpose the writer made use of the arrangement shown in fig. 2. The surfaces of a Ritchie Wedge, W, consisting of fine unglazed cardboard, were illuminated by the two heterochromatic sources of light to be studied, A and B. In front of the wedge a convex lens, L, is mounted on springs and attached to a cord which passes round the pin P mounted eccentrically on the pulley of the electric motor M. In front of the lens was placed a small black screen S pierced with an aperture 5 cms. high and 2 cms. wide.

If, now, the motor is stationary, the observer balances the brightness of the red and green by the equality of brightness method. But when the motor is set in rotation a band of flicker is produced between the coloured surfaces, since the dividing line between them appears to oscillate to and fro. By altering the position of the pin P the width of this band of flicker can be adjusted until it occupies exactly half the field of view, *i. e.* has the same dimensions as either of the coloured surfaces. We thus have the exact equivalent of observing two coloured surfaces of equal size, either simultaneously, when we judge their intensity by the equality of brightness method, or in rapid succession, when we judge their intensity by the absence of flicker. A typical result of varying the distance of the eye from the screen is shown

Fig. 2.

Arrangement of Rood Type of Photometer.

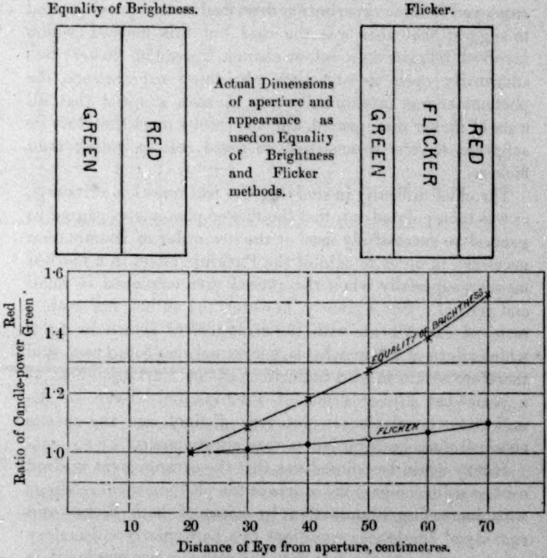


in fig. 3, the readings, with the eye at a distance of 20 cms., being taken as unity in each case.

The results of experiments on this matter were perplexing and variable in many respects. But one point brought out by this figure was invariably recognized. The effect of altering the distance of the eye was always much more marked in the case of the equality of brightness instrument.

Fig. 3.

Comparison of "Yellow-spot Effect" in Flicker and Equality of Brightness Photometers.



The discussion of this result will be postponed till a later stage in this paper.

It may be noted that Wild[*] has also stated that in some experiments undertaken by him, the effect of varying the region of the retina upon the apparent brightness of two heterochromatic surfaces was confirmed, at least qualitatively; but no such effect could be traced to exist in the case of the flicker photometer.

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[*] Electrician, Aug. 16, 1907, letter.

Flicker Photometers and the Purkinje Effect.

In the paper read before the Physical Society in 1906, the author referred to the question whether or no flicker instruments were, like those of the equality of brightness type, subject to the Purkinje effect. The results of a method employed in some experiments described in that paper seemed to suggest that this was the case, but this method (which involved reliance on a colour change instead of flicker) was admittedly open to objection. In these experiments, the photometer was intentionally run at such a speed that all trace of flicker disappeared, and the results must therefore be supposed to refer to sensation of fused colours rather than flicker.

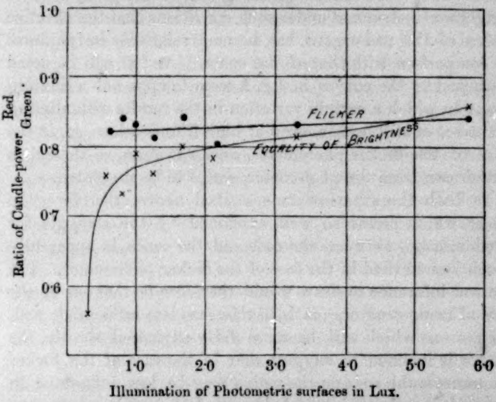
The chief difficulty in studying the real question at issue is, as was then pointed out, that the flicker photometer cannot in general be successfully used at the low order of illumination necessary in order to exhibit the Purkinje effect in a marked manner, especially when the retinal area employed is small and central. Subsequently, however, the author has made a series of experiments with the arrangement shown in fig. 2 which allows of a somewhat larger retinal area being used, and therefore seems to give indications of the Purkinje effect at a somewhat higher order of illumination. Even so the indications available are not very distinct, and the results obtained must be regarded as very approximate.

It may again be pointed out, that the arrangement utilized enables us to compare the course of the photometrical readings with increasing illumination by means of both flicker and equality of brightness sensations in a particularly satisfactory manner, owing to the fact that the retinal areas employed in each case can be kept practically the same: all that is necessary is to take a series of readings by the equality of brightness method and then, by setting the motor in rotation, to produce the proper-sized band of flicker.

Although, as stated above, very approximate, these readings certainly seem to warrant the suggestion that the Purkinje effect cannot be observed to the same extent in the case of the flicker sensation. A typical result is that shown in fig. 4.

Fig. 4.

Comparison of Purkinje Effect for Flicker and Equality of Brightness Photometers.



It may be added that other observers have recorded similar experiences. For instance, in the discussion following a paper by Dr. Steinmetz before the Illuminating Engineering Society of New York in December 1906, Mr. P. S. Millar referred to some tests on a mercury arc-lamp, which were carried out on a long photometrical bench with different distances between the sources compared and with a corresponding range of variation in the illumination of the photometer-screen. A Lummer-Brodhun photometer was first used, and it was then found that the candle-power of the mercury-vapour lamp, compared with a carbon-filament standard, apparently changed gradually, as the illumination of the photometer-screen was reduced, to a value higher by 30 per cent. than that at normal illuminations. But when a flicker photometer was used this difference did not occur.

Dicussion of Results shown in Figs. 3 *and* 4.

Let us now examine the results described in the two previous sections in greater detail. The main result of both experiments is that the Purkinje and Yellow-spot phenomena seem to be very much weaker when the flicker method is used. The flicker curve in fig. 3, in fact, seems to resemble that which is obtained by using an ordinary equality of brightness instrument under such conditions that the effective value of the rod-organs has become relatively insignificant in comparison with that of the cones. And it will be noted that, just as the curves in fig. 1 seem to approach a limiting shape in which a certain variation in the results with altering retinal area still occurs, even at high illuminations, so, in the case of the flicker photometer, *some* variation, as the eye is withdrawn from the photometer, seems to be suggested.

In both the experiments described above, therefore, the effect which seems so well explained by the struggle for predominance between the rods and the cones, is apparently much less marked in the case of the flicker photometer. The natural inference to draw would therefore be that *one* of the sets of competing organs is in this case less influential; and, for reasons which will be more fully explained shortly, the author is inclined to suggest that in the case of the flicker photometer the *rods* on the retina may be less influential in determining the point of balance. In an extreme case we might even suppose that the position of balance of an equality of brightness instrument is determined by combined rod and cone vision, while in the case of a flicker instrument it might be affected by cone-vision only.

The rods, as we have seen, are believed to differ essentially from the cones as regards light and colour-perception. Now the author is inclined to suggest that they may also be credited with a difference in the time taken to record and preserve a luminous impression. If, for instance, it were found that the speed of the flicker photometer, as ordinarily operated, was adjusted to the persistence of vision of the cones, but was too high for that of the rods, the above phenomena would be fairly well accounted for. Under these circumstances, we should obtain a flicker-sensation due to the cones and, superimposed over this, an impression of luminosity only, due to the fusion of rod-impulses, which might be supposed to succeed each other too rapidly to produce any flicker sensation. The rod-impulses therefore might not, under these circumstances, materially affect the place of balance of the photometer.

This would explain why the Purkinje effect was relatively insignificant, and why, with a flicker instrument, the yellow-spot effect, as shown in fig. 3, was less marked. In addition it might throw some light upon the fact that flicker instruments, however effective at ordinary high illuminations, do seem very difficult to use satisfactorily when the illumination is very low.

The appearance of the field of view of a photometer of the type shown in fig. 2, as seen under certain conditions, also seems to throw some light on the suggested behaviour of the rods. At low illuminations something like the following experience may sometimes be noticed. When comparing red and green, we may set the photometer so that the coloured flickerless surfaces on either side of the band of flicker appear about equally bright. The band of flicker in between will then be grey in colour, and probably, at the correct speed, the flicker can just be made to disappear.

If now the photometer is moved towards the red light the flicker reappears, and at the same time the red surface on one side of the intermediate band of flicker becomes obviously brighter than the green surface on the other ; the flicker band meanwhile changes from grey to a reddish tinge, and soon also becomes unmistakably brighter than the adjacent green.

But if we move the photometer towards the green surface we sometimes have a different experience. We again observe a distinct increase in brightness of the green, and we notice, as before, that the central grey band becomes green in tint and brighter than the adjacent red surface. But this gain in brightness is not accompanied by flicker; apparently the only effect is to superimpose a *steady* luminous impression produced by rod-impulses which succeed each other too rapidly to produce a flicker-sensation.

In short, it appears as though the speed which suited the production of flicker due to the red illumination was too high to do so in the case of the green illumination.

The effect (which, however, was only found to occur at very low illuminations, presumably on the borderland between rod- and cone-vision, and was not always easy to reproduce) seems to afford additional evidence in favour of the suggestion that the speed which is adapted to the luminous impressions received through the cones may be sometimes too high for those received through the rods.

It might, however, be well to guard against one possible source of misunderstanding as to the suggestion implied above regarding the Purkinje effect. In suggesting that the flicker photometer is less subject to the Purkinje effect than an equality of brightness one, the author does not mean to convey that it is impossible to demonstrate the profound change in retinal conditions responsible for this effect by a flicker method.

On the above supposition, it is only the struggle for predominance between the rods and cones that seems to be affected, owing to the difficulty of finding a speed to suit both types of organs. But there seems no doubt that, if the speed is sufficiently slow, a variety of flicker can exist at an order of illumination when we should be justified, apparently, in concluding that the rods are the main instruments of vision. Naturally, therefore, any flicker phenomena produced at such illuminations must be connected with the peculiarities of rod-vision, and may therefore even be used to show the existence of the Purkinje effect. For instance, it is easy to show that a pure red surface appears black and gives no flicker under such conditions. Haycraft, indeed (Proc. Roy. Soc. Lond. vol. lxi. 1897), has even used a purely flicker method to demonstrate the difference in the spectrum-curve of luminosity obtained at high and low illuminations, and thus, in a sense, to exhibit the Purkinje effect as this term is sometimes understood. But this result was naturally to be expected, and does not seem inconsistent with the suggestion the author has sought to convey.

The Theory of Rod-flicker and Cone-flicker.

It has been suggested that the means by which vision is effected through the rods may differ essentially from those utilized in the cone-organs, and it would therefore not be surprising to find that there was a distinct difference in the time for which a luminous impression could be retained in the two cases. Thus the visual purple which occurs in the rods, but not in the cones, has long been supposed to be the basis of a photochemical process by which the former organs achieve vision. Since no such substance seems to have yet been isolated in the cones (though Edridge Green has recently pointed out that the visual purple seems to exist *between* them)*, it would not be unreasonable to suppose that they retained images for a different length of time.

Let us therefore term the sensation received by using the portion of the retina where rods predominate "rod-flicker," and that received through using the part of the retina where cones predominate "cone-flicker."

A very interesting confirmation of the suggestion put forward above, seems to be supplied by the valuable work of Dr. T. C. Porter †. This observer found that, for illuminations varying from 0·25 lux up to 12,800 times this value, the connexion between the critical speed of a sector-disk (*i. e.* the speed at which the flicker caused by its rotation could just be made to vanish), and the illumination was of the form

$$n = k \, . \log \mathrm{I} + p,$$

where n = the number of revolutions per second when flicker vanishes, k and p are constants, and I is the illumination of the disk.

Now for illuminations below 0·25 lux a similar relation was found to hold, but the constant, k, suddenly diminished to half its former value and the curve connecting n and log I took an abrupt turn. This means that, at illuminations below this value, the duration of time during which a luminous image was retained appreciably unaltered in intensity was much greater, and the critical speed proportionately less.

Now it will be recalled that the above order of illumination, 0·25 lux, is just about the value at which, as we have seen earlier, the cones seem to cease action abruptly and cone-vision is replaced by rod-vision. It therefore seems highly probable that the abrupt change in the value of k corresponds to the substitution of rod-vision for cone-vision, and that the sensation of flicker tends to disappear at a lower speed in the case of the rods than in the case of the cones. This, indeed, seems in accordance with the observations of von Kries and other physiologists that the visual purple, with which rod-vision is believed to be connected, possesses a certain visual inertia in virtue of which it reacts appreciably later to an impulse than do the organs which are believed to be predominant at higher illuminations.

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* 'Illuminating Engineer' (London), March 1909, p. 210.

† Proc. Roy. Soc. London, vol. lxx. p. 315 (1902).

While mentioning this point, it may be of interest to refer to some other physiological results which seem to bear out this suggestion. The manner in which a luminous impression received by the eye disappears when the stimulus is withdrawn is, of course, of considerable interest as regards the theory of the flicker photometer. The matter has been made the subject of much study among physiologists, and the process of dying away of an impression seems to be of a very complicated character. Even bearing in mind the different behaviour of the rods and cones, this might be anticipated, but there are other contributory causes of complication.

It may be supposed that the impression received by the cones would be the first to weaken, and therefore the duration of undiminished impression, on which the point of balance of a flicker photometer presumably depends, mainly depends upon a purely cone-effect. But at a later stage the rod-impression would presumably begin to die away were it allowed to do so.

The theory of the "recurrent image" investigated by Shelford Bidwell in this country, von Kries in Germany, and others, is very interesting in this connexion. According to these observers, if a luminous surface, seen against a dark background, is moved before the eyes, a second ghostly image may, under suitable conditions, sometimes be seen, following the main image at a constant distance away. This image might presumably be explained by the later dying away of the rod impression presupposed above.

Now the recurrent image has been found by many observers not to occur when the central portion of the retina (where there are only cones) is used in vision, to be most easily seen by greenish light, and to be entirely absent when red light is used. All these facts strengthen the belief that this recurrent image is due to the rods, for, let it be remembered, there are believed to be no rods in the centre of the retina ; the rods are also most sensitive to greenish-blue light, and practically insensitive to red light (except under very exceptional conditions).

The subject of after-images is admittedly very complicated and not to be explained on the basis of the rod-action alone. Even here, however, the experiences of Prof. G. J. Burch *, referred to previously seem to furnish possible evidence in favour of the view that an image received through the rods tends to be retained for a longer time than one received through the cones. This investigator finds that, after remaining in a dark room for several hours, all the after-images constituting "dazzle" disappear. During the period of disappearance, it is found that coloured images disappear in a regular succession, red being the first colour to vanish and violet the last. This seems suggestive, for these colours are at the ends of the spectrum to which the rods are least and most sensitive respectively.

Admitting, therefore, that there is a certain amount of evidence in favour of the suggestion that the rods retain an impression in an undiminished form longer than the cones, and have therefore a lower critical speed, let us now deal with several other facts which support the suggestion that the rod-flicker can be regarded as distinct in other respects from the cone-flicker.

In the first place there seems to be, not only a difference in the critical speed of disappearance, but also *in character*. Several physiologists have pointed out that two varieties of flicker seem to exist. At low speeds the flicker-sensation is of a violent, coarse nature, but, as the speed is increased, this becomes modified and eventually gives way to a fine trembling variety, and this it is which we make use of, as a rule, in the flicker photometer.

If the speed is increased still further this, too, eventually disappears as all the impressions succeed each other too rapidly for flicker of any kind to be seen. From some observations the author is inclined to associate the violent variety of flicker with the rods and the finer variety with the cones. It seems that the violent variety is the only kind which can be properly seen when the illumination is very low, so that presumably only the rods would be in action. Under these conditions we observe the usual peculiarities connected with rod-vision. For instance, the violent flicker is best seen by averted gaze, so as to bring into action the peripheral region of the retina where the rods are most numerous. (This circumstance seems to have led Dr. Porter to make an ingenious suggestion. He supposes that in prehistoric ages the peripheral region of the retina so developed as to be more sensitive to flicker than the central region (although other senses are there most acute), in order to enable men to gain immediate intimation of any moving object in the neighbourhood such as might constitute a possible source of danger.)

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* Proc. Roy. Soc. London, 1905.

At high illuminations, however, when the speed is adjusted to a value such that the *fine variety* of flicker is produced, the exact opposite may be the case, *i. e.* we may find it possible to observe the flicker by the aid of the central region of the retina when it cannot be seen peripherally. Very interesting charts of the sensitiveness of different regions of the retina can be produced at different speeds. The author, however, found that, at high illuminations, the flicker, with increasing speed, tends to disappear last near the centre of the retina. This therefore favours the suggestion that this fine flicker, such as occurs at the higher speeds, is best seen by the aid of the cone organs.

[*Note.*—In this connexion the following experiment (which was shown on the occasion of the reading of this paper) is of interest. Two disks of equal size are prepared, and can be driven from the same motor, side by side, with the same speed. One of these disks is provided with red and black alternate sectors, the other with the same number of green and black sectors.

The two disks are illuminated with a very weak intensity and are driven at a slow speed. If the illumination is very weak indeed the red sectors appear to the eye black and the green sectors white, in accordance with the Purkinje effect. Under these conditions we naturally cannot perceive any flicker at all in the disk with red and black sectors.

At a somewhat brighter but still weak illumination, however, the red sectors will be visible, and we shall then see a flicker with both disks. But there is one striking difference in the conditions under which the flicker is best seen in the two cases.

In the case of the green and black disk we find that when we look straight at the disk we see no flicker; but by looking at it obliquely we immediately see the curious "wobbly" violent flicker which, it has been suggested, is mainly connected with the part of the retina where the rods predominate. In the case of the red and black disk the exact converse is the case. So long as we can see a flicker at all we can perceive it best by looking straight at the disk, *i. e.* by using the central region of the retina. But when we look at the disk obliquely the flicker becomes less apparent and, at a certain order of illumination, vanishes entirely. In addition the quality of the flicker is of a distinctly less violent kind and rather resembles the fine variety which is utilized in the flicker photometer.

It may be added that the order of illumination most favourable for the exhibition of the effect with the green and black disk is somewhat lower than that necessary for the red and black disk. A certain amount of care is thus necessary in order to exhibit both effects side by side successfully.

This experiment serves to show that the central part of the retina, where cones predominate, and the peripheral portion where there are chiefly rods, differ noticeably in the conditions under which flicker is best seen, and encourage the suggestion that there is something distinct about the nature of the flicker in the two cases.]

We may therefore imagine that, as the speed of a flicker is gradually increased from zero, the following changes occur. First the flicker is mainly of the violent kind associated with the rods, which tends to mask the perception of the fine variety; this is best seen by averted gaze. Next with increasing speed the violent flicker tends to fuse into a continuous impression of luminosity, and is ultimately replaced by the fine flicker which requires a higher speed to cause it to disappear. Lastly this too disappears. The impulses received through *both* sets of light-perceiving organs may now be supposed to follow so rapidly as to produce a continuous luminous impression.

A fact recorded by Porter and other observers that the critical speed of disappearance of flicker seems to depend upon the angle subtended at the eye by the flickering surface (*i. e.*, on the portion of the retina on which the image is received), would seem to be very conveniently explained by this view of the action of the two sets of organs.

An interesting point has been raised by Krüss *, which seems at first sight against the view that the action of a flicker photometer at high illuminations is mainly based on the behaviour of the cones. This observer points out that, in comparing say a Hefner lamp with an incandescent mantle by means of a flicker instrument, we notice that, when the speed is low and the photometer out of balance, fluctuations both in light and colour occur. As the speed is increased to what may be termed the working range the colours fuse into one another, but we still observe a flicker due to fluctuations in brightness, and by the aid of this we balance the photometer. This seems at first sight inconsistent with the supposition that the action of the instrument is based upon the cones with which the perception of both light and colour are associated. But it must be recalled that the colours of the two sources named are very impure; we are really comparing not red and green, but white lights slightly tinged with red and green respectively. Now the writer has found that, as the colours of the lights which it is sought to compare become more pronounced, so does the range of speed over which a flicker instrument can be used with success become narrowed down. In the case of very widely divergent colours there is really only one particular speed at which the colour-flicker has just disappeared, and the loss of sensitiveness due to the fusion of the light-impulses has not begun, at which the instrument can be used with

* *Journ. f. Gasbeleuchtung*, xxiv. June 16, 1907, p. 513.

accuracy. In short, the fusions of the colour- and light-impulses, when fairly pure colours are used, begin at just about the same speed; this, therefore, if anything would seem to favour the view that the action of the instrument so used is mainly dependent on the impulses received through the cones.

The portion of the retina employed in a flicker photometer is usually restricted to the central region, where there are comparatively few rods, and therefore these organs have probably not much to do with its action. Partly on this account, but chiefly because the correct speed for a flicker photometer used to compare widely different heterochromatic lights is so limited, it does not seem likely that information as to the theory of flicker photometers can readily be obtained by studying the effect of speed on the actual readings of the instrument. It is true that Lauriol* has been quoted as having recorded very marked variations in the reading of a flicker photometer at different speeds; but on closer examination it appears that this observer's results establish rather the difference between readings of a flicker photometer and a photometer of the equality of brightness pattern. For the difference recorded is that between the readings of a flicker instrument when used as such and when used with the flicker-wheel motionless.

The writer has sought to determine whether the permissible range of speed of a flicker instrument used to compare moderately heterochromatic sources could be responsible for any marked error, but without detecting any distinct variations due to this effect.

Colour-blind Observers and the Flicker Photometer.

Another possible direction in which an inquiry into the theory of the flicker photometer might be pushed is the examination of people having defective colour-vision. Some data are available on this subject, but they lead to somewhat uncertain conclusions. It appears, however, that in *some* cases colour-blind individuals have been observed to obtain correspondingly abnormal results with both flicker and equality of brightness photometers *, and it would certainly lead one to doubt the soundness of the principle underlying such instruments were this not the case.

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* *Bull. Soc. Ins. des Electriciens,* 1904.

In reality there seem to be so many different types of colour-blindness, and the existing knowledge of the subject is still so incomplete, that it is difficult for anyone but a physiologist to draw any valid conclusions on the theory of the flicker photometer by an examination of colour-blind individuals. The study of the principles underlying this instrument must, it would seem, be conducted in the light of the modern theories of the action of the retina. Physiologists, however, seem to have found some difficulty in reconciling many forms of colour-blindness with the usual theory of the action of the rods and cones. As an illustration of the complexities of the subject it may be mentioned that according to Edridge Green † some forms of colour-blindness arise, not through imperfect retinal apparatus, but through confusion at the centres of perception of colour in the brain; the same observer states that colour-blindness frequently accompanies some forms of insanity.

Concluding Remarks.

In conclusion the writer would like to say that he is conscious that the data presented are in many respects incomplete, and that other and more adequate experiments are needed before any decision as to the correctness of the suggestions brought forward can be taken. It may be noted, for instance, that these results described were, like those described in previous papers, obtained for the author's eye only. For this reason they are strictly comparable with the latter and in some respects seem very consistent. But naturally in order to be conclusive experiments in photometry should be based upon the observations of a large number of individuals, and upon a number of different types of instruments.

These experiments were mainly carried out at the Central Technical College, South Kensington, during 1906 and 1907, and the author had hoped to make the series of researches described in this paper somewhat more complete. Circumstances having prevented this, however, it occurred to him that their present publication might lead others to investigate the phenomena described more closely from the standpoint of photometry; the need for gaining some insight into the physiological facts underlying the flicker photometer may, perhaps, give these experiments a certain suggestive value.

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* See Brodhun, Gas World, Feb. 15, 1908; Dow, Electrician, Feb. 1 & 8, 1907; Tufts, Phys. Review, Dec. 1907, &c.

† 'Colour-Blindness and Colour-Perception.'

The writer is also well aware that the problem investigated is so essentially physiological in character as to be strictly only capable of being adequately studied by the combined efforts of the physiologist and the physicist interested in the problem from the photometrical side. It may be stated, too, that many points in connexion with the theory of the action of the rods and cones on the retina (only the main essentials of which are referred to in this paper), seem to be still the subject of much discussion among physiologists to whom we must look for the exact interpretation of their bearing on problems in photometry. Dr. Edridge Green, for instance, in a recent lecture before the Optical Society*, expressed his dissent from the theory of vision based on the supposed behaviour of the rods and cones, and expressed the view that visual impulses are only received through the latter organs. But the general phenomena on which the theory mainly rests, and which have been utilized in this paper, seem to be well authenticated.

ABSTRACT.

In this paper the Author suggests that something may be learned regarding the physiological phenomena governing the flicker photometer by observing whether it is subject to certain physiological effects, such as the "yellow spot" and "Purkinje" phenomena, which are a cause of uncertainty in heterochromatic photometry undertaken with instruments of the ordinary "Equality of Brightness" class.

Some experiments are described which show that the effects referred to do occur, but are apparently much less marked; the question arises why this should be the case. The Author suggests a possible explanation based on the assumption that the rod-elements on the retina, in addition to the distinguishing peculiarities attributed to these organs as regards the perception of light and colour, also differ from the "cones" in the

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* Optical Society (London) Oct. 21, 1909.

fact that they seem to receive a luminous impression more slowly, and retain it longer than these organs; in other words the "last" of an undiminished luminous impression is longer.

This additional peculiarity is of little consequence in an ordinary photometer of the equality of brightness type, but may possibly play a part in the flicker instrument; it seems to explain why certain effects should be more clearly perceived in one case than in the other.

According to this theory we may imagine the flicker-effect to be due to two distinct portions, received by the agency of the rods and cones respectively. Under certain conditions the speed of a flicker photometer may be supposed to be suitable for the use of "cone-flicker" but too high for the "rod-flicker," which becomes fused into a steady luminous impression and thus does not affect the readings of the instrument.

Some other physiological effects, such as the existence of a recurrent image, seem in harmony with the above view. These matters are, however, still the subject of much dispute between physiologists, and the Author regards his experiments as being essentially of a suggestive character requiring more detailed examination. In addition, it is urged that one must be cautious in seeking to draw deductions from cases of colour-blindness, as many different varieties of this affliction exist; thus in some cases colour-blindness is apparently attributable to some disturbance of the centre of perception in the brain, rather than to any peculiarity of the retina.

DISCUSSION.

Dr. **EDRIDGE GREEN** expressed his interest in the paper and stated that he was prepared to agree with the effects described, although he thought they were capable of explanation by another theory. The theory of Von Kries was relied upon because the rods contained visual purple while the cones did not. Light falling on the eye liberates visual purple, photochemical action is set up and nerve sensation results. The speaker described the theory he favoured for the explanation of colour-blindness.

Mr. **A. P. TROTTER**, after referring to the doubtful value of the flicker photometer for industrial purposes, said that the present investigation should be considered as a scientific one. In discussing the effect of varying the distance of the photometer from the eye, Mr. Dow was really dealing with the angle subtended or the angle of the field of view, and it would be better to use this method of expressing the relation, for it could then be referred to definite parts of the retina. Sir W. Abney and others had mapped the areas of the retina and had found boundaries of colour perception. Abney considers the angles subtended at the centre of curvature of the retina, and his smallest boundary lies within about 15 degrees from the centre, or, according to usual convention, a field of view of about 30 degrees. The whole of Mr. Dow's present experiments fall well within this, and do not extend to the peripheral parts of the retina. Taking the width of the Joly blocks as 2 cm., his starting point 10 cm. is equivalent to a field of view of about 11 degrees, and his 40 cm. to about 2½ degrees. For the purpose of investigating the action of the peripheral parts of the retina, a photometrical device covering a large field of view should be used, and the middle part should be blocked out.

Dr. **Russell** suggested that possibly mechanical vibrations are set up in the rods and cones of the eye. These oscillations having probably different periods might explain some of the curious phenomena that Mr. Dow had shown to the meeting. The cones in the eye did not satisfy the mathematical definition of a cone, and it would be extremely difficult to make any calculations. He was not convinced that the rods were unable to perceive colour. According to the theory advanced at threshold illumination only the rods are active. Dr. Burch has shown that after the eye is rested its sensitiveness to the more refrangible rays is greatly increased. In this case, at threshold illumination, when presumably only the rods are active, the sodium lines appear to be green. The rods therefore at low illuminations perceive green, and so the assumption that they do not perceive colour is inadmissible.

Mr. C. C. **Paterson** expressed his great interest in the paper, and appreciation of the experimental skill shown by the author. He said he wished to make some observations on the so-called yellow-spot effect, because, whilst being able to offer no well substantiated theory to account for the phenomena which Mr. Dow had observed, he could not altogether make the application of the rod and cone theory in this instance agree with the experimental facts. Considering the interesting lower diagram on page 87, there was a curve there which showed that a 40 per cent. difference was obtained in the contrast between two surfaces illuminated by a red and a green light respectively, according to whether the observer's eye was 20 or 60 centimetres away from the surfaces. The author explained this difference by assuming that in one case more of the retina was employed than in the other, so that in the 20 centimetre case a ring of the retina surrounding the yellow spot and containing more rods, was being used. Referring to the top left-hand patch in Figure 3, it might be noted that at 20 centimetres distance a square which just included the words 'red' and 'green' was the area which approximately covered the yellow spot, if the eye were directed on the centre of the figure. At 60 centimetres distance the whole height of the figure about covered the yellow spot. If Mr. Dow's explanation were correct it meant, as far as he could see, that as he obtained a balance and looked at the centre of his patches from 20 centimetres distance, the author must have been conscious of at least a 40 per cent. difference between the comparison at the centre and the comparison at the top or bottom of the two illuminated areas. It was difficult to understand how balance could be obtained if this condition existed, and Mr. Trotter's suggestion of blocking out the centre of the patch and observing only the boundaries seemed to be a valuable one. He did not think that in making comparisons the area of the retina outside the yellow spot was used to any appreciable extent. All who were in the habit of using photometers were probably conscious that if they kept the eye fixed, only a small area of the illuminated surfaces was being considered by the observer at one moment, and the sensation from the outlying area of the surface had no appreciable weight compared with that from the centre, on to which the eye was directed. He (Mr. Paterson) was inclined to believe that many of the discrepancies of photometric comparisons were due as much, if not more, to psychological than to physiological causes. He did not see why the retina should be regarded as the only faulty link in the chain. In the latter portion of the paper in which the flicker question was discussed, the Author showed the observed facts to be very well explained by the theory of the rods and cones, and he congratulated him upon the flicker experiment which he had shown.

Mr. **Rayner** referred to the use of a Ritchie wedge as a flicker photometer, and asked if there was a greater intensity at the edges of the field due to some direct reflection.

Mr. J. S. Dow, replying to Dr. Edridge Green's remarks, said that he quite recognized that the theory of the action of the retina was a matter for physiologists to deal with, and he was much interested in Dr. Edridge Green's explanation of the most recent view. His suggestions regarding the flicker photometer were intended to bring these phenomena into line with the well authenticated other physiological effects which the rod and cone theory had served to explain so completely. He quite appreciated the importance of Dr. Edridge Green's discovery of visual purple in the fovea. Mr. Trotter's suggestion regarding expressing results in angles subtended at the eye was a good one: in leaving the results in their present form he had merely wished to give an idea of conditions occurring in an actual photometer. For experimental purposes, however, it was often desirable to use a larger retinal area. Mr. Trotter's suggestion that the centre part of the retina should be blocked out was diametrically opposite to that of M. Blondel, who had suggested using *only* the centre part of the retina, where cones only exist. As a principle in photometry, however, it seemed desirable not to depart further from the condition of the eye in practice than could be avoided. Dr. Russell had criticised the suggestion that the rods could not distinguish colour but yet were most sensitive to blue-green light. As he understood the theory, it seemed to be suggested that the perception of "luminosity" might be greatest for light of this wave-length, although the rods could not analyse colour, *i. e.* all qualities of light appeared to them white. Dr. Burch's experiments, however, had apparently shown that after resting in the dark for two hours or more, it was possible to distinguish colours as long as any light was visible at all; this result certainly seemed at variance with the above view of the behaviour of the rods and in itself suggested that the theory required modification. In reply to Mr. Paterson's point, he might say that in viewing a coloured surface he was not conscious of any want of uniformity in brightness due to retinal peculiarities: the eye seemed to weigh up the appearance as a whole, just as in the case of white light. No doubt psychological influences played some part in all these problems: however, the existence of the yellow spot and Purkinje effects had been known for a long time and both effects had been authenticated by many observers. It was only the *explanation* that seemed still a matter for conjecture.

VIII. *Tables of the* ber *and* bei *and* ker *and* kei *Functions, with further Formulæ for their Computation. By* HAROLD G. SAVIDGE[*].

SHORTLY after the publication of Dr. A. Russell's paper on the ber and bei and allied functions (Phil. Mag. April 1909), Professor Lees suggested that tables of the ker and kei functions, as defined by Dr. Russell, and tables of composite functions would be of great use to electricians and others; and the subjoined tables are the outcome of this suggestion. Other suggestions have been given in the course of the work by Professor Lees and Dr. Russell, for which the author desires to take this opportunity of expressing his thanks.

Definitions.

The functions tabulated may be defined as follows:—

$$\operatorname{ber} x + \iota \operatorname{bei} x = I_0(x\sqrt{\iota})\dagger = J_0(\iota x\sqrt{\iota}),$$

$$\operatorname{ber} x - \iota \operatorname{bei} x = J_0(x\sqrt{\iota}) = I_0(\iota x\sqrt{\iota}),$$

$$\operatorname{ker} x + \iota \operatorname{kei} x = K_0(x\sqrt{\iota}) = Y_0(\iota x\sqrt{\iota}),$$

$$\operatorname{ker} x - \iota \operatorname{kei} x = Y_0(x\sqrt{\iota}) = K_0(\iota x\sqrt{\iota}),$$

$$X(x) = \operatorname{ber}^2 x + \operatorname{bei}^2 x,$$

$$V(x) = \operatorname{ber}'^2 x + \operatorname{bei}'^2 x,$$

$$Z(x) = \operatorname{ber} x \operatorname{ber}' x + \operatorname{bei} x \operatorname{bei}' x,$$

$$W(x) = \operatorname{ber} x \operatorname{bei}' x - \operatorname{bei} x \operatorname{ber}' x.$$

$X_1(x)$, $V_1(x)$, etc. are the corresponding ker and kei forms; thus

$$X_1(x) = \operatorname{ker}^2 x + \operatorname{kei}^2 x, \text{ etc.}$$

$$S(x) = \operatorname{ber}' x \operatorname{ker}' x + \operatorname{bei}' x \operatorname{kei}' x,$$

$$T(x) = \operatorname{bei}' x \operatorname{ker}' x - \operatorname{ber}' x \operatorname{kei}' x.$$

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[*] Read November 12, 1909.

[†] See Gray and Matthews, 'Bessel's Functions.' J and Y are the Bessel functions of the first and second kind respectively, and I and K the same functions for unreal values of the argument. Dr. Russell's function $Y(x)$ is here called $V(x)$. ber′ x and bei′ x stand for the differential coefficients of ber x and bei x with respect to x.

The series α, β, α', β' are also tabulated. They are useful for interpolation, and β appears in Dr. Russell's formula (107) (there called ϵ).

They may be defined by the equations :

$$\operatorname{ber} x = \frac{e^{\alpha}}{\sqrt{2\pi x}} \cos \beta, \qquad \operatorname{bei} x = \frac{e^{\alpha}}{\sqrt{2\pi x}} \sin \beta,$$

and

$$\operatorname{ker} x = \sqrt{\frac{\pi}{2x}}\, e^{\alpha'} \cos \beta', \quad \operatorname{kei} x = \sqrt{\frac{\pi}{2x}}\, e^{\alpha'} \sin \beta',$$

where x is not less than 6, and

$$\alpha = \frac{x}{\sqrt{2}} + \frac{1}{8\sqrt{2}x} - \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} \cdots\cdots$$

$$\beta = \frac{x}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} - \frac{25}{384\sqrt{2}x^3} \cdots\cdots$$

$$\alpha' = -\frac{x}{\sqrt{2}} - \frac{1}{8\sqrt{2}x} + \frac{25}{384\sqrt{2}x^3} - \frac{13}{128x^4} \cdots\cdots$$

$$\beta' = -\frac{x}{\sqrt{2}} - \frac{\pi}{8} + \frac{1}{8\sqrt{2}x} - \frac{1}{16x^2} + \frac{25}{384\sqrt{2}x^3} \cdots\cdots$$

When x is small, we have

$$\alpha = \tfrac{1}{2} \log \{2\pi x \mathrm{X}(x)\}, \qquad \beta = \arctan (\operatorname{bei} x / \operatorname{ber} x),$$

$$\alpha' = \tfrac{1}{2} \log \left\{ \frac{2x\mathrm{X}_1(x)}{\pi} \right\}, \qquad \beta' = \arctan (\operatorname{kei} x / \operatorname{ker} x).$$

Further Formulæ (x greater than 5).

Since $\operatorname{ber} x + i \operatorname{bei} x = \mathrm{I}_0(x\sqrt{\iota})$, we can find the values of $\operatorname{ber} x$ and $\operatorname{bei} x$ by putting $x\sqrt{\iota}$ for x in

$$\mathrm{I}_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 + \frac{1^2}{8x} + \frac{1^2 . 3^2}{|\underline{2} . (8x)^2} + \frac{1^2 . 3^2 . 5^2}{|\underline{3} . (8x)^3} + \ldots \right\}$$

(Gray & Matthews : Bessel's Functions, p. 68.)

Making this substitution, noticing that

$$\sqrt[4]{\iota} = \tfrac{1}{2}\{\sqrt{2 + \sqrt{2}} + \iota\sqrt{2 - \sqrt{2}}\},$$

and equating the real and imaginary parts we obtain

$$\operatorname{ber} x=\frac{e^{x/\sqrt{2}}}{2\sqrt{2\pi x}}\left\{\left(\sqrt{2+\sqrt{2}}\cos\frac{x}{\sqrt{2}}+\sqrt{2-\sqrt{2}}\sin\frac{x}{\sqrt{2}}\right)r\right.$$
$$\left.+\left(\sqrt{2+\sqrt{2}}\sin\frac{x}{\sqrt{2}}-\sqrt{2-\sqrt{2}}\cos\frac{x}{\sqrt{2}}\right)s\right\},$$

$$\operatorname{bei} x=\frac{e^{x/\sqrt{2}}}{2\sqrt{2\pi x}}\left\{\left(\sqrt{2+\sqrt{2}}\sin\frac{x}{\sqrt{2}}-\sqrt{2-\sqrt{2}}\cos\frac{x}{\sqrt{2}}\right)r\right.$$
$$\left.-\left(\sqrt{2+\sqrt{2}}\cos\frac{x}{\sqrt{2}}+\sqrt{2-\sqrt{2}}\sin\frac{x}{\sqrt{2}}\right)s\right\},$$

where

$$r=1+\frac{1^2}{8\sqrt{2}x}-\frac{1^2.3^2.5^2}{\underline{|3}(8x)^3\sqrt{2}}-\frac{1^2.3^2.5^2,7^2}{\underline{|4}(8x)^4}-\frac{1^2.3^2.5^2.7^2.9^2}{\underline{|5}(8x)^5\sqrt{2}}+\ldots$$

$$s=\frac{1^2}{8\sqrt{2}x}+\frac{1^2.3^2}{\underline{|2}(8x)^2}+\frac{1^2.3^2.5^2}{\underline{|3}(8x)^3\sqrt{2}}-\frac{1^2,3^2.5^2.7^2,9^2}{\underline{|5}(8x)^5\sqrt{2}}-\ldots$$

These expressions reduce to

$$\operatorname{ber} x=f\cos\left(\frac{x}{\sqrt{2}}-\phi\right),$$

$$\operatorname{bei} x=f\sin\left(\frac{x}{\sqrt{2}}-\phi\right),$$

where

$$f=\frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\left\{1+\frac{1}{8\sqrt{2}x}+\frac{1}{256x^2}-\frac{133}{2048\sqrt{2}x^3}-\frac{27690}{512^2x^4}\cdots\right\}$$

$$\tan\phi=\sqrt{2}-1+\frac{\sqrt{2}-1}{4x}+\frac{12-5\sqrt{2}}{64x^2}+\frac{20\sqrt{2}-13}{256x^3}+\frac{435\sqrt{2}-560}{8192x^4}\cdots$$

From the above we obtain the following formulæ :

$$X(x)=\frac{e^{x\sqrt{2}}}{2\pi x}\left\{1+\frac{1}{4\sqrt{2}x}+\frac{1}{64x^2}-\frac{33}{256\sqrt{2}x^3}-\frac{3594}{128^2x^4}\cdots\right\}$$

$$V(x)=\frac{e^{x\sqrt{2}}}{2\pi x}\left\{1-\frac{3}{4\sqrt{2}x}+\frac{9}{64x^2}+\frac{75}{256\sqrt{2}x^3}+\frac{4950}{128^2x^4}\cdots\right\}$$

$$Z(x)=\frac{e^{x\sqrt{2}}}{2\pi x}\left\{\frac{1}{\sqrt{2}}-\frac{3}{8x}-\frac{15}{64\sqrt{2}x^2}-\frac{45}{512x^3}+\frac{630}{128^2\sqrt{2}x^4}\cdots\right\}$$

$$W(x)=\frac{e^{x\sqrt{2}}}{2\pi x}\left\{\frac{1}{\sqrt{2}}+\frac{1}{8x}+\frac{9}{64\sqrt{2}x^2}+\frac{39}{512x^3}+\frac{150}{128^2\sqrt{2}x^4}\cdots\right\}$$

$$\operatorname{ber}' x=\frac{e^{\eta}}{\sqrt{2\pi x}}\cos\omega,$$

$$\operatorname{bei}' x=\frac{e^{\eta}}{\sqrt{2\pi x}}\sin\omega,$$

where

$$r = \frac{x}{\sqrt{2}} - \frac{3}{8\sqrt{2}x} + \frac{21}{128\sqrt{2}x^3} + \frac{27}{128x^4} \cdots$$

$$\omega = \frac{x}{\sqrt{2}} + \frac{\pi}{8} + \frac{3}{8\sqrt{2}x} + \frac{3}{16x^2} + \frac{21}{128\sqrt{2}x^3} \cdots$$

There are similar formulæ throughout for the ker and kei and composite functions:

$$X_1(x) = \frac{\pi}{2xe^{x\sqrt{2}}} \left\{1 - \frac{1}{4\sqrt{2}x} + \frac{1}{64x^2} + \frac{33}{256\sqrt{2}x^3} - \frac{3594}{128^2x^4} \cdots \right\}$$

and similar formulæ for $V_1(x)$, $Z_1(x)$, and $W_1(x)$. $Z_1(x)$ and $W_1(x)$ may be found by means of the tables for $Z(x)$ and $W(x)$ and the formulæ :—

$$Z(x)Z_1(x) = -\frac{1}{8x^2}\left(1 - \frac{3}{4x^2}\cdots\right)$$

$$W(x)W_1(x) = -\frac{1}{8x^2}\left(1 + \frac{1}{4x^2}\cdots\right).$$

In these two series the coefficient of x^{-4} is zero. They can be used when x is not less than 6 : for smaller values of the argument these functions are tabulated

$$\left.\begin{aligned} S(x) &= c \sin x\sqrt{2} - d \cos x\sqrt{2} \\ T(x) &= -d \sin x\sqrt{2} - c \cos x\sqrt{2} \end{aligned}\right\}$$

where

$$c = \frac{3}{8\sqrt{2}x^2} + \frac{75}{512\sqrt{2}x^4} - \frac{15435}{256^2\sqrt{2}x^6} \cdots$$

$$d = \frac{1}{2x} - \frac{9}{128x^3} + \frac{2475}{128^2x^5} \cdots$$

and these may be written

$$S(x) = -g \cos \psi$$

$$T(x) = -g \sin \psi,$$

where

$$g=\frac{1}{2x}+\frac{27}{128x^5}\cdots\cdots$$

$$\psi=x\sqrt{2}+\frac{3}{4\sqrt{2}x}+\frac{21}{64\sqrt{2}x^3}-\frac{1899}{2560\sqrt{2}x^5}\cdots\cdot$$

x.		ber x.		bei x.		ker x.		kei x.
1	+	$10^{-1}\times 9{\cdot}844$	+	$10^{-1}\times 2{\cdot}496$	+	$10^{-1}\times 2{\cdot}867$	−	$10^{-1}\times 4{\cdot}950$
2	+	$10^{-1}\times 7{\cdot}517$	+	$10^{-1}\times 9{\cdot}723$	−	$10^{-2}\times 4{\cdot}166$	−	$10^{-1}\times 2{\cdot}024$
3	−	$10^{-1}\times 2{\cdot}214$	+	$1{\cdot}938$	−	$10^{-2}\times 6{\cdot}703$	−	$10^{-2}\times 5{\cdot}112$
4	−	$2{\cdot}563$	+	$2{\cdot}293$	−	$10^{-2}\times 3{\cdot}618$	+	$10^{-3}\times 2{\cdot}198$
5	−	$6{\cdot}230$	+	$10^{-1}\times 1{\cdot}160$	−	$10^{-2}\times 1{\cdot}151$	+	$10^{-2}\times 1{\cdot}119$
6	−	$8{\cdot}858$	−	$7{\cdot}335$	−	$10^{-4}\times 6{\cdot}531$	+	$10^{-3}\times 7{\cdot}216$
7	−	$3{\cdot}633$	−	$10\times 2{\cdot}124$	+	$10^{-3}\times 1{\cdot}922$	+	$10^{-3}\times 2{\cdot}700$
8	+	$10\times 2{\cdot}097$	−	$10\times 3{\cdot}502$	+	$10^{-3}\times 1{\cdot}486$	+	$10^{-4}\times 3{\cdot}696$
9	+	$10\times 7{\cdot}394$	−	$10\times 2{\cdot}471$	+	$10^{-4}\times 6{\cdot}372$	−	$10^{-4}\times 3{\cdot}192$
10	+	$10^2\times 1{\cdot}388$	+	$10\times 5{\cdot}637$	+	$10^{-4}\times 1{\cdot}295$	−	$10^{-4}\times 3{\cdot}075$
11	+	$10^2\times 1{\cdot}330$	+	$10^2\times 2{\cdot}572$	−	$10^{-5}\times 4{\cdot}779$	−	$10^{-4}\times 1{\cdot}495$
12	−	$10^2\times 1{\cdot}285$	+	$10^2\times 5{\cdot}470$	−	$10^{-5}\times 6{\cdot}308$	−	$10^{-5}\times 3{\cdot}899$
13	−	$10^2\times 8{\cdot}827$	+	$10^2\times 6{\cdot}466$	−	$10^{-5}\times 3{\cdot}474$	+	$10^{-6}\times 5{\cdot}387$
14	−	$10^3\times 2{\cdot}131$	−	$10^2\times 1{\cdot}609$	−	$10^{-5}\times 1{\cdot}088$	+	$10^{-5}\times 1{\cdot}268$
15	−	$10^3\times 2{\cdot}967$	−	$10^3\times 2{\cdot}953$	−	$10^{-8}\times 1{\cdot}514$	+	$10^{-6}\times 7{\cdot}963$
16	−	$10^2\times 6{\cdot}595$	−	$10^3\times 8{\cdot}191$	+	$10^{-6}\times 2{\cdot}466$	+	$10^{-6}\times 2{\cdot}895$
17	+	$10^3\times 9{\cdot}484$	−	$10^4\times 1{\cdot}309$	+	$10^{-6}\times 1{\cdot}797$	+	$10^{-7}\times 2{\cdot}861$
18	+	$10^4\times 3{\cdot}096$	−	$10^3\times 7{\cdot}454$	+	$10^{-7}\times 7{\cdot}438$	−	$10^{-7}\times 4{\cdot}555$
19	+	$10^4\times 5{\cdot}625$	+	$10^4\times 2{\cdot}804$	+	$10^{-7}\times 1{\cdot}293$	−	$10^{-7}\times 3{\cdot}982$
20	+	$10^4\times 4{\cdot}749$	+	$10^5\times 1{\cdot}148$	−	$10^{-8}\times 7{\cdot}715$	−	$10^{-7}\times 1{\cdot}859$
21	−	$10^4\times 7{\cdot}616$	+	$10^5\times 2{\cdot}337$	−	$10^{-8}\times 8{\cdot}636$	−	$10^{-8}\times 4{\cdot}388$
22	−	$10^5\times 4{\cdot}155$	+	$10^5\times 2{\cdot}539$	−	$10^{-8}\times 4{\cdot}535$	+	$10^{-8}\times 1{\cdot}097$
23	−	$10^5\times 9{\cdot}536$	−	$10^5\times 1{\cdot}527$	−	$10^{-8}\times 1{\cdot}320$	+	$10^{-8}\times 1{\cdot}824$
24	−	$10^6\times 1{\cdot}242$	−	$10^6\times 1{\cdot}460$	+	$10^{-10}\times 8{\cdot}786$	+	$10^{-8}\times 1{\cdot}083$
25	+	$10^3\times 9{\cdot}798$	−	$10^6\times 3{\cdot}809$	+	$10^{-9}\times 3{\cdot}723$	+	$10^{-9}\times 3{\cdot}703$
26	+	$10^6\times 4{\cdot}936$	−	$10^6\times 5{\cdot}744$	+	$10^{-9}\times 2{\cdot}532$	+	$10^{-10}\times 1{\cdot}912$
27	+	$10^7\times 1{\cdot}489$	−	$10^6\times 2{\cdot}308$	+	$10^{-10}\times 9{\cdot}915$	−	$10^{-10}\times 7{\cdot}257$
28	+	$10^7\times 2{\cdot}553$	+	$10^7\times 1{\cdot}578$	+	$10^{-10}\times 1{\cdot}367$	−	$10^{-10}\times 5{\cdot}791$
29	+	$10^7\times 1{\cdot}825$	+	$10^7\times 5{\cdot}695$	−	$10^{-10}\times 1{\cdot}320$	−	$10^{-10}\times 2{\cdot}563$
30	−	$10^7\times 4{\cdot}612$	+	$10^8\times 1{\cdot}100$	−	$10^{-10}\times 1{\cdot}294$	−	$10^{-11}\times 5{\cdot}290$
∞		∞		∞		0		0

x.	α.	β.	$-\alpha'$.	$-\beta'$.
1	·9343	·2483	·7844	1·0458
2	1·4717	·9126	1·4560	1·7738
3	2·1362	1·6846	2·1499	2·4900
4	2·8473	2·4119	2·8501	3·2023
5	3·5532	3·1230	3·5529	3·9127
6	4·2572	3·8332	4·2574	4·6221
7	4·9622	4·5430	4·9623	5·3310
8	5·6678	5·2520	5·6678	6·0394
9	6·3737	5·9606	6·3737	6·7475
10	7·0799	6·6689	7·0799	7·4555
11	7·7862	7·3769	7·7862	8·1633
12	8·4926	8·0848	8·4926	8·8710
13	9·1992	8·7925	9·1992	9·5786
14	9·9058	9·5001	9·9058	10·2862
15	10·6125	10·2077	10·6125	10·9937
16	11·3192	10·9152	11·3192	11·7011
17	12·0260	11·6227	12·0260	12·4085
18	12·7328	12·3301	12·7328	13·1159
19	13·4397	13·0375	13·4397	13·8232
20	14·1465	13·7449	14·1465	14·5306
21	14·8534	14·4522	14·8534	15·2379
22	15·5604	15·1595	15·5604	15·9452
23	16·2673	15·8668	16·2673	16·6524
24	16·9742	16·5741	16·9742	17·3597
25	17·6812	17·2813	17·6812	18·0669
26	18·3882	17·9886	18·3882	18·7742
27	19·0951	18·6958	19·0951	19·4814
28	19·8021	19·4031	19·8021	20·1886
29	20·5091	20·1103	20·5091	20·8958
30	21·2161	20·8175	21·2161	21·6030
∞	∞	∞	∞	∞

The following simple expressions for α and β give a four figure accuracy when x is not greater than 1·5 and 1·4, respectively:

$$\alpha = \tfrac{1}{2}\log 2\pi + \tfrac{1}{2}\log x + \frac{1}{4}\left(\frac{x}{2}\right) - \frac{11}{192}\left(\frac{x}{2}\right)^{8} + \frac{473}{25920}\left(\frac{x}{z}\right)^{12} \ldots$$

(obtained by logarithmic expansion), $\tfrac{1}{2}\log 2\pi = \cdot 918939$;

$$\beta = \left(\frac{x}{2}\right)^2 - \frac{1}{9}\left(\frac{x}{2}\right)^6 + \frac{19}{600}\left(\frac{x}{2}\right)^{10} \dots$$

$\left(\text{obtained by expansion of arctan } \frac{\text{bei } x}{\text{ber } x}\right)$.

x.	$X(x)$.	$V(x)$.	$Z(x)$.	$W(x)$.
0	1·000	0	0	0
1	1·031	$10^{-1}\times 2{\cdot}513$	$10^{-2}\times 6{\cdot}266$	$10^{-1}\times 5{\cdot}052$
2	1·510	1·084	$10^{-1}\times 5{\cdot}209$	1·169
3	3·803	3·240	2·054	2·847
4	$10\times 1{\cdot}183$	$10\times 1{\cdot}097$	6·909	8·446
5	$10\times 3{\cdot}883$	$10\times 3{\cdot}375$	$10\times 2{\cdot}345$	$10\times 2{\cdot}757$
6	$10^{2}\times 1{\cdot}323$	$10^{2}\times 1{\cdot}177$	$10\times 8{\cdot}215$	$10\times 9{\cdot}393$
7	$10^{2}\times 4{\cdot}643$	$10^{2}\times 4{\cdot}203$	$10^{2}\times 2{\cdot}943$	$10^{2}\times 3{\cdot}294$
8	$10^{3}\times 1{\cdot}666$	$10^{3}\times 1{\cdot}526$	$10^{3}\times 1{\cdot}072$	$10^{3}\times 1{\cdot}181$
9	$10^{3}\times 6{\cdot}077$	$10^{3}\times 5{\cdot}621$	$10^{3}\times 3{\cdot}953$	$10^{3}\times 4{\cdot}305$
10	$10^{4}\times 2{\cdot}245$	$10^{4}\times 2{\cdot}093$	$10^{4}\times 1{\cdot}474$	$10^{4}\times 1{\cdot}590$
11	$10^{4}\times 8{\cdot}383$	$10^{4}\times 7{\cdot}863$	$10^{4}\times 5{\cdot}541$	$10^{4}\times 5{\cdot}935$
12	$10^{5}\times 3{\cdot}157$	$10^{5}\times 2{\cdot}977$	$10^{5}\times 2{\cdot}099$	$10^{5}\times 2{\cdot}234$
13	$10^{6}\times 1{\cdot}197$	$10^{6}\times 1{\cdot}134$	$10^{5}\times 7{\cdot}999$	$10^{5}\times 8{\cdot}472$
14	$10^{6}\times 4{\cdot}568$	$10^{6}\times 4{\cdot}344$	$10^{6}\times 3{\cdot}065$	$10^{6}\times 3{\cdot}233$
15	$10^{7}\times 1{\cdot}752$	$10^{7}\times 1{\cdot}672$	$10^{7}\times 1{\cdot}180$	$10^{7}\times 1{\cdot}240$
16	$10^{7}\times 6{\cdot}752$	$10^{7}\times 6{\cdot}461$	$10^{7}\times 4{\cdot}561$	$10^{7}\times 4{\cdot}777$
17	$10^{8}\times 2{\cdot}612$	$10^{8}\times 2{\cdot}506$	$10^{8}\times 1{\cdot}770$	$10^{8}\times 1{\cdot}848$
18	$10^{9}\times 1{\cdot}014$	$10^{8}\times 9{\cdot}752$	$10^{8}\times 6{\cdot}887$	$10^{8}\times 7{\cdot}175$
19	$10^{9}\times 3{\cdot}950$	$10^{9}\times 3{\cdot}806$	$10^{9}\times 2{\cdot}688$	$10^{9}\times 2{\cdot}794$
20	$10^{10}\times 1{\cdot}543$	$10^{10}\times 1{\cdot}489$	$10^{10}\times 1{\cdot}052$	$10^{10}\times 1{\cdot}091$
21	$10^{10}\times 6{\cdot}041$	$10^{10}\times 5{\cdot}842$	$10^{10}\times 4{\cdot}127$	$10^{10}\times 4{\cdot}273$
22	$10^{11}\times 2{\cdot}371$	$10^{11}\times 2{\cdot}296$	$10^{11}\times 1{\cdot}622$	$10^{11}\times 1{\cdot}677$
23	$10^{11}\times 9{\cdot}326$	$10^{11}\times 9{\cdot}044$	$10^{11}\times 6{\cdot}390$	$10^{11}\times 6{\cdot}596$
24	$10^{12}\times 3{\cdot}675$	$10^{12}\times 3{\cdot}568$	$10^{12}\times 2{\cdot}521$	$10^{12}\times 2{\cdot}599$
25	$10^{13}\times 1{\cdot}451$	$10^{13}\times 1{\cdot}410$	$10^{12}\times 9{\cdot}966$	$10^{13}\times 1{\cdot}026$
26	$10^{13}\times 5\ 736$	$10^{13}\times 5{\cdot}582$	$10^{13}\times 3{\cdot}945$	$10^{13}\times 4{\cdot}057$
27	$10^{14}\times 2{\cdot}271$	$10^{14}\times 2{\cdot}213$	$10^{14}\times 1{\cdot}564$	$10^{14}\times 1{\cdot}606$
28	$10^{14}\times 9{\cdot}007$	$10^{14}\times 8{\cdot}783$	$10^{14}\times 6{\cdot}207$	$10^{14}\times 6{\cdot}370$
29	$10^{15}\times 3{\cdot}576$	$10^{15}\times 3{\cdot}490$	$10^{15}\times 2{\cdot}467$	$10^{15}\times 2{\cdot}529$
30	$10^{16}\times 1{\cdot}422$	$10^{16}\times 1{\cdot}389$	$10^{16}\times 9{\cdot}799$	$10^{16}\times 1\ 005$
∞	∞	∞	∞	∞

We may write $V(x)$, $Z(x)$, $W(x)$ in a form similar to Dr. Russell's formula $X(x)=\frac{e^{2a}}{2\pi x}$, namely:

$$V(x)=\frac{e^{2\eta}}{2\pi x}, \text{ where } \eta=\frac{x}{\sqrt{2}}-\frac{3}{8\sqrt{2}x}+\frac{21}{128\sqrt{2}x^3}+\frac{27}{128x^4}.$$

x.	$V/X^{(x)}$.	$Z/X^{(x)}$.	$W/X^{(x)}$.	$Z/V^{(x)}$.	$W/V^{(x)}$.
0	0	0	0	0	∞
1	·2437	·06076	·4899	·2494	2·0104
2	·7177	·3449	·7738	·4806	1·0782
3	·8518	·5399	·7485	·6339	·8787
4	·8512	·5842	·7141	·6863	·8389
5	·8691	·6040	·7101	·6950	·8171
6	·8901	·6211	·7101	·6978	·7979
7	·9051	·6339	·7094	·7004	·7838
8	·9162	·6433	·7088	·7021	·7736
9	·9250	·6505	·7084	·7033	·7659
10	·9321	·6562	·7081	·7040	·7597
11	·9380	·6609	·7079	·7046	·7547
12	·9430	·6648	·7078	·7050	·7506
13	·9472	·6681	·7077	·7054	·7471
14	·9509	·6709	·7076	·7056	·7442
15	·9541	·6734	·7075	·7058	·7418
16	·9569	·6755	·7075	·7060	7394
17	·9593	·6774	·7074	·7061	·7374
18	·9615	·6791	·7074	·7062	·7357
19	·9635	·6805	·7074	7063	·7342
20	·9653	·6819	·7073	·7064	·7328
21	·9669	·6831	·7073	·7065	·7315
22	·9684	·6842	·7073	·7065	·7304
23	·9698	·6852	·7073	·7066	·7293
24	·9710	·6861	·7073	·7066	·7284
25	·9721	·6870	·7073	·7067	·7275
26	·9732	·6877	·7072	·7067	·7267
27	·9742	·6885	·7073	·7067	·7260
28	·9751	·6891	·7072	·7068	·7253
29	·9759	·6898	·7072	·7068	·7247
30	·9767	·6903	·7072	·7068	·7241
∞	1	·7071	·7071	·7071	·7071

$$\mathrm{Z}(x) = \frac{e^{2\zeta}}{2\pi x}, \text{ where } \zeta = \frac{x}{\sqrt{2}} - \frac{3}{8\sqrt{2}x} - \frac{3}{16x^2} - \frac{27}{128\sqrt{2}x^3} - \frac{9}{128x^4}.$$

$$\mathrm{W}(x) = \frac{e^{2\omega}}{2\pi x}, \text{ where } \omega = \frac{x}{\sqrt{2}} + \frac{1}{8\sqrt{2}x} + \frac{1}{16x^2} + \frac{23}{384\sqrt{2}x^3} - \frac{1}{128x^4}.$$

x.	$\mathrm{X}_1(x)$.	$\mathrm{V}_1(x)$.	$\mathrm{S}(x)$.		$\mathrm{T}(x)$.	
0	∞	∞		0		$-0{\cdot}5$
1	$10^{-1}\times 3{\cdot}272$	$10^{-1}\times 6{\cdot}066$	+	$10^{-1}\times 2{\cdot}186$	−	$10^{-1}\times 3{\cdot}235$
2	$10^{-2}\times 4{\cdot}270$	$10^{-2}\times 5{\cdot}968$	+	$10^{-1}\times 2{\cdot}541$	+	$10^{-2}\times 1{\cdot}063$
3	$10^{-3}\times 7{\cdot}106$	$10^{-3}\times 8{\cdot}933$	+	$10^{-2}\times 4{\cdot}733$	+	$10^{-1}\times 1{\cdot}634$
4	$10^{-3}\times 1{\cdot}314$	$10^{-3}\times 1{\cdot}563$	−	$10^{-1}\times 1{\cdot}104$	+	$10^{-2}\times 5{\cdot}949$
5	$10^{-4}\times 2{\cdot}577$	$10^{-4}\times 2{\cdot}962$	−	$10^{-2}\times 6{\cdot}254$	−	$10^{-2}\times 7{\cdot}801$
6	$10^{-5}\times 5{\cdot}250$	$10^{-5}\times 5{\cdot}901$	+	$10^{-2}\times 5{\cdot}501$	−	$10^{-2}\times 6{\cdot}263$
7	$10^{-5}\times 1{\cdot}099$	$10^{-5}\times 1{\cdot}214$	+	$10^{-2}\times 6{\cdot}086$	+	$10^{-2}\times 3{\cdot}741$
8	$10^{-6}\times 2{\cdot}344$	$10^{-6}\times 2{\cdot}560$	−	$10^{-2}\times 2{\cdot}347$	+	$10^{-2}\times 5{\cdot}793$
9	$10^{-7}\times 5{\cdot}078$	$10^{-7}\times 5{\cdot}491$	−	$10^{-2}\times 5{\cdot}421$	−	$10^{-2}\times 1{\cdot}217$
10	$10^{-7}\times 1{\cdot}113$	$10^{-7}\times 1{\cdot}195$	+	$10^{-3}\times 2{\cdot}910$	−	$10^{-2}\times 4{\cdot}992$
11	$10^{-8}\times 2{\cdot}464$	$10^{-8}\times 2{\cdot}628$	+	$10^{-2}\times 4{\cdot}521$	−	$10^{-3}\times 4{\cdot}684$
12	$10^{-9}\times 5{\cdot}500$	$10^{-9}\times 5{\cdot}833$	+	$10^{-2}\times 1{\cdot}087$	+	$10^{-2}\times 4{\cdot}022$
13	$10^{-9}\times 1{\cdot}236$	$10^{-9}\times 1{\cdot}305$	−	$10^{-2}\times 3{\cdot}506$	+	$10^{-2}\times 1{\cdot}582$
14	$10^{-10}\times 2{\cdot}792$	$10^{-10}\times 2{\cdot}936$	−	$10^{-2}\times 1{\cdot}967$	−	$10^{-2}\times 2{\cdot}981$
15	$10^{-11}\times 6{\cdot}341$	$10^{-11}\times 6{\cdot}646$	+	$10^{-2}\times 2{\cdot}456$	−	$10^{-2}\times 2{\cdot}254$
16	$10^{-11}\times 1{\cdot}446$	$10^{-11}\times 1{\cdot}512$	+	$10^{-2}\times 2{\cdot}451$	+	$10^{-2}\times 1{\cdot}939$
17	$10^{-12}\times 3{\cdot}311$	$10^{-12}\times 3{\cdot}452$	−	$10^{-2}\times 1{\cdot}438$	+	$10^{-2}\times 2{\cdot}566$
18	$10^{-13}\times 7{\cdot}608$	$10^{-13}\times 7{\cdot}912$	−	$10^{-2}\times 2{\cdot}607$	−	$10^{-3}\times 9{\cdot}593$
19	$10^{-13}\times 1{\cdot}753$	$10^{-13}\times 1{\cdot}820$	+	$10^{-3}\times 5{\cdot}085$	−	$10^{-2}\times 2{\cdot}582$
20	$10^{-14}\times 4{\cdot}051$	$10^{-14}\times 4{\cdot}197$	+	$10^{-2}\times 2{\cdot}498$	+	$10^{-4}\times 9{\cdot}118$
21	$10^{-15}\times 9{\cdot}383$	$10^{-15}\times 9{\cdot}704$	+	$10^{-3}\times 2{\cdot}883$	+	$10^{-2}\times 2{\cdot}363$
22	$10^{-15}\times 2{\cdot}178$	$10^{-15}\times 2{\cdot}250$	−	$10^{-2}\times 2{\cdot}185$	+	$10^{-3}\times 6{\cdot}261$
23	$10^{-16}\times 5{\cdot}068$	$10^{-16}\times 5{\cdot}226$	−	$10^{-3}\times 9{\cdot}195$	−	$10^{-2}\times 1{\cdot}970$
24	$10^{-16}\times 1{\cdot}181$	$10^{-16}\times 1{\cdot}216$	+	$10^{-2}\times 1{\cdot}726$	−	$10^{-2}\times 1{\cdot}166$
25	$10^{-17}\times 2{\cdot}757$	$10^{-17}\times 2{\cdot}836$	+	$10^{-2}\times 1{\cdot}366$	+	$10^{-2}\times 1{\cdot}461$
26	$10^{-18}\times 6{\cdot}447$	$10^{-18}\times 6{\cdot}625$	−	$10^{-2}\times 1{\cdot}182$	+	$10^{-2}\times 1{\cdot}517$
27	$10^{-18}\times 1{\cdot}510$	$10^{-18}\times 1{\cdot}550$	−	$10^{-2}\times 1{\cdot}621$	−	$10^{-3}\times 8{\cdot}948$
28	$10^{-19}\times 3{\cdot}540$	$10^{-19}\times 3{\cdot}631$	+	$10^{-3}\times 6{\cdot}073$	−	$10^{-2}\times 1{\cdot}679$
29	$10^{-20}\times 8{\cdot}312$	$10^{-20}\times 8{\cdot}517$	+	$10^{-2}\times 1{\cdot}693$	+	$10^{-3}\times 3{\cdot}253$
30	$10^{-20}\times 1{\cdot}954$	$10^{-20}\times 2{\cdot}000$	−	$10^{-4}\times 3{\cdot}196$	+	〃 $^{-2}\times 1{\cdot}666$
∞	0	0		0		0

Similarly for $V_1(x)$, $Z_1(x)$, and $W_1(x)$, e. g.

$$V_1(x)=\frac{\pi e^{2\eta'}}{2x},\text{ where }\eta'=-\frac{x}{\sqrt{2}}+\frac{3}{8\sqrt{2x}}-\frac{21}{128\sqrt{2x^3}}+\frac{27}{128x^4},\text{ etc.}$$

x.	$-Z_1(x)$.	$-W_1(x)$.
1	$10^{-1}\times 3{\cdot}736$	$10^{-1}\times 2{\cdot}428$
2	$10^{-2}\times 4{\cdot}005$	$10^{-2}\times 3{\cdot}073$
3	$10^{-3}\times 6{\cdot}145$	$10^{-3}\times 5{\cdot}072$
4	$10^{-3}\times 1{\cdot}086$	$10^{-4}\times 9{\cdot}343$
5	$10^{-4}\times 2{\cdot}071$	$10^{-4}\times 1{\cdot}829$

IX. *The Effective Resistance and Inductance of a Helical Coil.*

By J. W. NICHOLSON, *M.A., D.Sc.*[*]

THE problem of the propagation of alternating currents in wires has been solved in but few cases. For a single wire isolated in space, a solution has been given by Sommerfeld[†]. If the return current be conducted along a concentric sheath, instead of by the dielectric, the problem is that treated earlier by Sir J. J. Thomson[‡]. The case in which the return current flows along a parallel wire has been discussed by Mie[§], and to a first approximation, in a simpler manner, by Morton[||], who has given, in other papers[¶], a detailed examination of the distribution of forces, and applied his method also to the case of a greater number of wires.

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[*] Read November 26, 1909.

[†] Wied. *Ann.* lxvii. p. 233 (1899).

[‡] Recent Researches, p. 262.

[§] *Ann. der Phys.* ii. p. 202 (1900).

[||] Phil. Mag. Dec. 1900.

[¶] Phil. Mag. May 1901, Sept. 1902, June 1903.

The influence of electrostatic capacity in wires renders the mathematical investigations very complicated, but from a practical point of view, the main object to be attained is a knowledge of the effective inductance and resistance of the wires when the capacity and leakage are not so great as to produce a sensible attenuation of the wave in a short distance.

The concentration of the current in the outer regions of the wires, when the frequency is high, completely changes the character of the inductance and resistance, and a knowledge of their values for steady currents is frequently of little service. But even with the restriction to wires with small capacity and leakage, mathematical solutions have not hitherto progressed far. The problem of the concentric main was discussed by Maxwell * and Heaviside †, and afterwards for very high frequencies by Lord Rayleigh ‡. A practical formula for the resistance of the inner conductor was given by Lord Kelvin §, whose solution is not essentially different from that of Heaviside.

Formulæ for the effective inductance in this case have also been obtained by Sir J. J. Thomson ‖, and an exhaustive treatment of the whole problem by more simple methods, in which the ranges of application of all the previous formulæ are critically examined, has been given by Russell ¶.

The investigation of the effective inductance and resistance of two parallel wires, one being the return of the other, was made by the author **, and developed, in a later paper ††, in a form capable of more immediate practical use. The upper limit necessary to the capacity was at the same time examined.

The main problem of importance in which the wires are bent is that of the single helical coil of considerable length. Solutions have hitherto dealt with the case in which the wire is closely wound round a circular cylinder, forming a helix whose pitch is very small. Wien [*] first discussed the problem, but assumed that the current distribution is the same at all points equidistant from the axis. Battelli showed that this assumption was not lawful, and the results do not agree with experiment. Shortly afterwards, Sommerfeld [†] investigated the problem in so far as the resistance is concerned, and his method was applied later by Coffin [‡] to a determination of inductance. Picciati [§] gave an independent solution, but like those of Sommerfeld and Wien, its agreement with experiment was not very good. The best solution in this respect is that of Cohen [||], which appears to be quite satisfactory for all cases in which the pitch of winding is very small.

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* Elec. and Mag. vol. ii. § 690.

† Electrical Papers, i. p. 353; ii. p. 64 *et seq.*

‡ Phil. Mag. xxii. p. 381 (1886); Scientific Papers, ii. p. 486.

§ Journ. Inst. Elec. Eng. xxiii. p. 4 (1889); Math. & Phys. Papers, iii. p. 491.

‖ Recent Researches, p. 295.

¶ Phil. Mag. April 1909; Proc. Phys. Soc. xxi.

** Phil. Mag. Feb. 1909; Proc. Phys. Soc. xxi.

†† Phil. Mag. Sept. 1909; Proc. Phys. Soc. xxii.

The present paper deals with the corresponding formulæ for a helical wire whose pitch is not very small, wound on a cylinder whose radius is large compared with that of the section of the wire, which is of course circular, and cannot be treated as square in the manner applied by Cohen to the other extreme case. A rigorous solution would be very difficult, but it is shown that some very simple approximate results are sufficient for practical purposes. The coil is to be regarded as sufficiently long for the effects of the ends to be neglected. A connexion between these formulæ and those of Cohen cannot be made without a consideration of the difficult intermediate case in which the pitch is moderately small.

Choice of Coordinates.

Let the axis of the cylinder on which the wire is wound be chosen as that of z in a Cartesian system. Defining the central curve as the locus of the centres of all normal sections of the wire, this curve will be a helix, and if a is the radius of the cylinder, and θ the twist in a plane perpendicular to z of the radius in that plane measured from a line parallel to x, the coordinates of a point on the helix become

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[*] *Ann. der Phys.* xiv. p. 1 (1904).

[†] *Ann. der Phys.* xv. p. 673 (1904).

[‡] Bulletin of Bureau of Standards, ii. p. 275.

[§] *Il Nuovo Cimento* (5) ii. p. 35.

[||] Bulletin of Bureau of Standards, iv. no. 1.

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = a\theta\tan\alpha \quad . \quad . \quad (1)$$

where α is the angle of the helix.

Let $(\rho\,\phi)$ be polar coordinates in the plane of the normal section at the point defined by θ. The initial line from which ϕ is to be measured is perpendicular to z and to the tangent of the helix. Thus if z were vertical, the initial line in any section would be the horizontal line. Now the direction cosines, at the point θ, of the tangent to the helix are

$$(-\sin\theta\cos\alpha, \quad \cos\theta\cos\alpha, \quad \sin\alpha).$$

A line perpendicular to this and to the axis of z $(0\,0\,1)$ is readily shown to have a direction

$$(\cos\theta, \quad \sin\theta, \quad 0).$$

This is the line $\phi=0$ in the section defined by θ.

Let (λ, μ, ν) be the direction cosines of the radius vector ρ in the section. It is at an angle ϕ to the above line, and is perpendicular to the central curve, so that

$$\lambda\cos\theta+\mu\sin\theta = \cos\phi,$$
$$-\lambda\sin\theta\cos\alpha+\mu\cos\theta\cos\alpha+\nu\sin\alpha = 0,$$

with $$\lambda^2+\mu^2+\nu^2 = 1.$$

Thus

$$\begin{aligned}1-\nu^2 = \lambda^2+\mu^2 &= (\lambda\cos\theta+\mu\sin\theta)^2+(\mu\cos\theta-\lambda\sin\theta)^2\\ &= \cos^2\phi+\nu^2\tan^2\alpha,\end{aligned}$$

or $\quad \nu = \pm\cos\alpha\sin\phi.$

We choose the positive sign, and deduce

$$\left.\begin{aligned}\lambda &= \cos\theta\cos\phi+\sin\alpha\sin\theta\sin\phi,\\ \mu &= \sin\theta\cos\phi-\sin\alpha\cos\theta\sin\phi,\\ \nu &= \cos\alpha\sin\phi\end{aligned}\right\} \quad . \quad (2)$$

as the direction of the radius vector.

The coordinates of any point of space are therefore

$$\left.\begin{aligned} x &= a\cos\theta+\rho\lambda \\ &=(a+\rho\cos\phi)\cos\theta+\rho\sin\alpha\sin\theta\sin\phi, \\ y &= a\sin\theta+\rho\mu \\ &=(a+\rho\cos\phi)\sin\theta-\rho\sin\alpha\cos\theta\sin\phi, \\ z &= a\theta\tan\alpha+\rho\nu \\ &= a\theta\tan\alpha+\rho\cos\alpha\sin\phi \end{aligned}\right\} \quad . \quad (3)$$

We shall call $(\rho\,\phi\,\theta)$ the *helical coordinates* of any point of space. If δs (which only involves $\delta\theta$) be an element of length perpendicular to a normal section at the point (ρ, ϕ, θ), then the space elements δs, $\delta\rho$, $\rho\delta\phi$ are mutually perpendicular, and the surfaces $\rho=$const. are those of helical wires with a common central curve. This system of coordinates is therefore appropriate to problems in which surface conditions must be satisfied at the boundaries of such wires.

Property of Orthogonal Systems.

In the problem here contemplated, in which an alternating electromotive force Ee^{ipt} is applied to the terminals of a long helical coil, the electric and magnetic vectors in and near the coil will not depend on θ, if the other portion of the circuit is kept at a distance. Thus $\partial/\partial\theta=0$.

Consider now a general orthogonal system in which the components of all vectors are independent of one coordinate. Denoting the three coordinates by $(\rho\,\phi\,\theta)$, let the corresponding space elements along their instantaneous directions of increase at any point be

$$(ds_1, ds_2, ds_3) = (d\rho/p_1, d\phi/p_2, d\theta/p_3). \quad . \quad . \quad (4)$$

Let (α, β, γ), (x, y, z) be the components of magnetic and electric force along these directions. Since the operation $\partial/\partial\theta$ annuls any component, the solenoidal conditions become.

$$\left.\begin{aligned} \frac{\partial}{\partial\rho}\frac{X}{p_2p_3}+\frac{\partial}{\partial\phi}\frac{Y}{p_1p_3} &= 0, \\ \frac{\partial}{\partial\rho}\frac{\alpha}{p_2p_3}+\frac{\partial}{\partial\phi}\frac{\beta}{p_1p_3} &= 0\cdot \end{aligned}\right\} \quad . \quad . \quad . \quad (5)$$

It is assumed also that $(p_1 p_2 p_3)$ are independent of θ, as is the case in all important applications. Functions χ and ψ may be introduced, where

$$\left.\begin{aligned} X &= p_2p_3\,\partial\chi/\partial\phi, \quad & Y &= p_1p_3\,\partial\chi/\partial\rho, \\ \alpha &= p_2p_3\,\partial\psi/\partial\phi, \quad & \beta &= p_1p_3\,\partial\psi/\partial\rho \end{aligned}\right\} \quad . \quad . \quad (6)$$

By Faraday's law, if μ be the permeability,

$$\begin{aligned} \frac{-\mu}{p_1p_3}\frac{\partial\gamma}{\partial t} &= \frac{\partial}{\partial\rho}\frac{Y}{p_2} - \frac{\partial}{\partial\phi}\frac{X}{p_1} \\ &= \frac{\partial}{\partial\rho}\frac{p_1p_2}{p_3}\frac{\partial\chi}{\partial\rho} + \frac{\partial}{\partial\phi}\frac{p_2p_3}{p_1}\frac{\partial\chi}{\partial\phi}\,; \quad . \quad . \quad (7) \end{aligned}$$

and by Ampère's law, in a dielectric,

$$\frac{1}{V^2p_2p_3}\frac{\partial X}{\partial t} = \mu\frac{\partial}{\partial\phi}\frac{\gamma}{p_3}\,; \quad . \quad . \quad . \quad . \quad (8)$$

whence, on reduction, χ, and similarly ψ, satisfy the equation

$$\frac{\partial}{\partial\rho}\frac{p_1p_3}{p_2}\frac{\partial\psi}{\partial\rho} + \frac{\partial}{\partial\phi}\frac{p_2p_3}{p_1}\frac{\partial\psi}{\partial\phi} = \frac{p_3}{V^2p_1p_2}\frac{\partial^2\psi}{\partial t^2}, \quad . \quad . \quad (9)$$

where V is the velocity of propagation of electromagnetic disturbances. Thus any disturbance in which the vectors are independent of one coordinate of an orthogonal system must be made up of two types, in which the electric and magnetic forces along the direction of increase of that coordinate are respectively zero everywhere.

If the wave-length of the oscillation, supposed simply periodic, be $2\pi/k$, the equation for ψ becomes

$$\frac{\partial}{\partial\rho}\frac{p_1p_3}{p_2}\frac{\partial\psi}{\partial\rho} + \frac{\partial}{\partial\phi}\frac{p_2p_3}{p_1}\frac{\partial\psi}{\partial\phi} + \frac{k^2p_3}{p_1p_2}\psi = 0, \quad . \quad (10)$$

and for a conducting medium, it is only necessary to change k^2. In the present problem the oscillation defined by ψ exists, but not that defined by χ. The vectors are given by

$$\begin{aligned} X &= Y = 0, & \alpha &= \gamma = 0, \\ Z &= -\frac{p_3}{V^2}\frac{\partial\psi}{\partial t}, & \beta &= -p_1p_3\frac{\partial\psi}{\partial\rho}. \quad . \quad . \quad (11) \end{aligned}$$

The electric force is along θ increasing, or parallel at any point to the corresponding tangent of the central curve. The magnetic force is along ϕ increasing. At the surface of any wire of the helical system, both forces are tangential, as they should be.

Solution as a Fourier Series.

If ds_1 is an element of arc along ρ increasing,

$$\left(\frac{\partial s_1}{\partial\rho}\right)^2=\left(\frac{\partial x}{\partial\rho}\right)^2+\left(\frac{\partial y}{\partial\rho}\right)^2+\left(\frac{\partial z}{\partial\rho}\right)^2=\lambda^2+\mu^2+\nu^2=1;$$

so that $p_1=1$. Moreover,

$$\left(\frac{\partial s_2}{\partial\phi}\right)^2=\left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2+\left(\frac{\partial z}{\partial\phi}\right)^2$$

$$=(-\rho\sin\phi\cos\theta+\rho\sin\alpha\sin\theta\cos\phi)^2$$

$$+(\rho\sin\phi\sin\theta+\rho\sin\alpha\cos\theta\cos\phi)^2+\rho^2\cos^2\alpha\cos^2\phi=\rho^2$$

on reduction, and $p_2=\rho^{-1}$. Similarly

$$p_3=\left\{a^2\sec^2\alpha+2a\rho\cos\phi+\rho^2(\cos^2\phi+\sin^2\alpha\sin^2\phi)\right\}^{-\frac{1}{2}}. \quad . \quad (12)$$

If the wire be thin, $p_3=a^{-1}\cos\alpha$, and we are led to the usual solution for the straight wire of great length.

The differential equation may be reduced to

$$\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi^2}+k^2\rho\psi+\rho\frac{\partial\psi}{\partial\rho}\frac{1}{p_3}\frac{\partial p_3}{\partial\rho}+\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\frac{1}{p_3}\frac{\partial p_3}{\partial\phi}=0. \quad . \quad (13)$$

Now

$$\frac{1}{p_3}\frac{\partial p_3}{\partial\phi}=\frac{a\rho\sin\phi+\rho^2\sin\phi\cos\phi\cos^2\alpha}{a^2\sec^2\alpha+2a\rho\cos\phi+\rho^2(\cos^2\phi+\sin^2\alpha\sin^2\phi)}$$

$$=\frac{\rho}{a}\sin\phi\cos^2\alpha-\frac{\rho^2}{2a^2}\sin2\phi\cos^4\alpha+\ldots\ldots \quad . \quad . \quad . \quad (14)$$

on expansion in powers of ρ/a, replacing the trigonometric functions by those of multiple angles. Again,

$$\frac{\rho}{p_3}\frac{\partial p_3}{\partial\rho}=-\frac{\rho}{a}\cos^2\alpha\cos\phi-\frac{\rho^2}{2a^2}\cos^2\alpha(2-3\cos^2\alpha-\cos^2\alpha\cos2\phi). \quad . \quad (15)$$

We shall not retain ρ^3/a^3 or higher powers.

The equation (13) has a solution of the form

$$\psi=A_0+A_1\cos\phi+A_2\cos2\phi+\ldots\ldots \quad . \quad . \quad (16)$$

where the functions A are independent of ϕ. By substitution, the accent denoting $\partial/\partial\rho$,

$$\left(\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+k^2\rho\right)(A_0+A_1\cos\phi+A_2\cos 2\phi+\ldots)-\frac{1}{\rho}(A_1\cos\phi+2^2A_2\cos 2\phi\ldots\ldots)$$

$$-\frac{\rho}{2a}\cos^2\alpha\{2\cos\phi A_0'+(1+\cos 2\phi)A_1'+(\cos\phi+\cos 3\phi)A_2'+\ldots\ldots\}$$

$$-\frac{\rho^2}{2a^2}\cos^2\alpha(2-3\cos^2\alpha)(A_0'+\cos\phi A_1'+\cos 2\phi A_2'+\ldots\ldots)$$

$$+\frac{\rho^2}{4a^2}\cos^4\alpha\{2\cos 2\phi A_0'+(\cos\phi+\cos 3\phi)A_1'+(1+\cos 4\phi)A_2'+\ldots\ldots\}$$

$$-\frac{\cos^2\alpha}{2a}\{(1-\cos 2\phi)A_1+2(\cos\phi-\cos 3\phi)A_2+3(\cos 2\phi-\cos 4\phi)A_3+\ldots\ldots\}$$

$$+\frac{\rho}{4a^2}\cos^4\alpha\{(\mathrm{ccs}\,\phi-\cos 3\phi)A_1+2(1-\cos 4\phi)A_2+3(\cos\phi-\cos 5\phi)A_3+\ldots\ldots\},$$

we deduce the following system of equations, where

$$D\equiv\frac{d}{d\rho}\rho\frac{d}{d\rho}+k^2\rho, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$DA_0-\frac{\rho}{2a}\cos^2\alpha A_1'-\frac{\rho^2}{2a^2}\cos^2\alpha(2-3\cos^2\alpha)A_0'+\frac{\rho^2}{4a^2}\cos^4\alpha A_2'$$
$$-\frac{A_1}{2a}\cos^2\alpha+\frac{\rho A_2}{2a^2}\cos^4\alpha=0,$$

$$DA_1-\frac{A_1}{\rho}-\frac{\rho}{2a}\cos^2\alpha(2A_0'+A_2')-\frac{\rho^2}{2a^2}\cos^2\alpha(2-3\cos^2\alpha)A_1'$$
$$+\frac{\rho^2}{4a^2}\cos^4\alpha(A_1'+A_3')-\frac{A_2}{a}\cos^2\alpha+\frac{\rho}{4a^2}\cos^4\alpha(A_1+3A_3)=0,. \quad (18)$$

and so on. But it is evident that A_0, A_1, A_2…… form a sequence, each member of which is of a higher order in a^{-1} than those before.

For the purpose of this investigation, A_0 is required to order ρ^2/a^2, and therefore, as an intermediary, A_1 to order ρ/a only, and A_2 not at all. Therefore the equations may be written more simply in the forms

$$\left.\begin{aligned} DA_0-\frac{\rho^2}{2a^2}\cos^2\alpha(2-3\cos^2\alpha)\frac{dA}{d\rho}&=\frac{\cos^2\alpha}{2a}\frac{d}{d\rho}(\rho A_1),\\ DA_1-\frac{A_1}{\rho}&=\frac{\cos^2\alpha}{a}\rho\frac{dA_0,}{d\rho} \quad . \quad . \quad . \quad . \end{aligned}\right\} (19)$$

and A_1 must be eliminated.

It is convenient to take a new variable $k\rho = x$, so that

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{1}{x^2}\right)A_1 = \frac{\cos^2\alpha}{ka}\frac{dA_0}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{\cos^2\alpha(2-3\cos^2\alpha)}{2k^2a^2}x\frac{d}{dx}\right)A_0 = \frac{\cos^3\alpha}{2kax}\frac{d}{dx}(xA_1). \; . \quad (21)$$

Now (20) may be written

$$A_1 + \frac{d}{dx}\frac{1}{x}\frac{d}{dx}xA_1 = \frac{\cos^2\alpha}{ka}\frac{dA_0}{dx},$$

or

$$\left(\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} + 1\right)\frac{1}{x}\frac{d}{dx}xA_2 = \frac{\cos^2\alpha}{kax}\frac{d}{dx}x\frac{dA_0}{dx},$$

and the elimination with (21) may be effected at once, yielding

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1\right)\left\{\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{\cos^2\alpha(2-3\cos^2\alpha)}{2k^2a^2}x\frac{d}{dx}\right\}A_0$$

$$= \frac{\cos^4\alpha}{2k^2a^2}\frac{1}{x}\frac{d}{dx}x\frac{dA_0}{dx}; \quad . \quad (22)$$

and it is at once evident that A_0 is only altered to the second order by the bending of the wire. It is henceforth supposed that not only a, but ka is large, as this case includes nearly all of practical interest. When ka is small, a special investigation is necessary.

Ignoring the effect of bending, the differential equation becomes

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1\right)^2 A_0 = 0;$$

so that, for a solution finite on the central curve,

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1\right)A_0 \propto J_0(x);$$

and finally

$$A_0 = AJ_0(x) + BxJ_1(x),$$

where A and B are arbitrary. In determining the self-induction and resistance, we only require the modification, inside the metal, of the form $J_0(x)$, and in the outer medium, of the form $K_0(\iota x)$. The solution $xJ_1(x)$ may be discarded, as it corresponds to nothing in the case of the straight wire, to which the formulæ must lead when a is infinite, or $\alpha=0$.

Writing therefore

$$A_0=J_0(x)+\frac{\cos^2\alpha}{2k^2a^2}u, \quad . \quad . \quad . \quad . \quad (23)$$

where u is a new variable, only required in its first term, not involving $(ka)^{-2}$, then on substitution, and rejection of $(ka)^{-2}$ in the result,

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1\right)^2u=\frac{\cos^2\alpha}{x}\frac{d}{dx}x\frac{dJ_0}{dx}+(2-3\cos^2\alpha)\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1\right)x\frac{dJ_0}{dx}.$$

By properties of the Bessel function of zero order, this becomes

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1\right)^2u=+(1-5\sin^2\alpha)J_0(x); \quad . \quad (24)$$

whence

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}+1\right)u=+\tfrac{1}{2}(1-5\sin^2\alpha)xJ_1(x),$$

ignoring the complementary function. Finally,

$$u=-\tfrac{1}{8}(1-5\sin^2\alpha)x^2J_0(x), \quad . \quad . \quad . \quad . \quad (25)$$

again ignoring a complementary function. The total form thus ignored is $AxJ_1(x)+BJ_0(x)$, of which the first term pertains to the redundant solution of the original equation. The second has no relation to the original $J_0(x)$ with unity as coefficient. We write therefore

$$A_0=\left\{1-\frac{\cos^2\alpha(1-5\sin^2\alpha)x^2}{16k^2a^2}\right\}J_0(x) \quad . \quad . \quad (26)$$

as the value of A_0 to the second order. Substitution in the differential equation directly verifies this solution, and it therefore corresponds to the function $J_0(x)$ of the unbent wire.

Solution for the Dielectric.

The above values hold for the interior of the wire of resistivity σ, provided that $k^2=-4\pi\mu\iota p/\sigma$, where $p/2\pi$ is the frequency of oscillation. For the dielectric, the appropriate function outside a straight wire would be $K_0(\iota h\rho)$, where $h=p/V$ takes the place of k. The function is defined by

$$K_n(\iota u)=\int_0^\infty d\phi \cosh n\phi\, e^{-\iota u \cosh\phi} \quad . \quad . \quad . \quad . \quad (27)$$

and possesses the properties

$$\left.\begin{aligned} \frac{d}{du}u^n K_n(u) &= -u^n K_{n-1}(u), \\ K_{n+1}(u)-K_{n-1}(u) &= \frac{2n}{u}K_n(u), \\ K_{n+1}(u)+K_{n-1}(u) &= -2\frac{dK_n(u)}{du}, \\ \frac{d}{du}K_0(u) &= -K_1(u). \end{aligned}\right\} \quad . \quad . \quad . \quad (28)$$

The analysis required in determining the corrected function is very similar to that above, and leads finally to the function

$$B_0(\iota h\rho)=\left\{1-\frac{\cos^2\alpha(1-5\sin^2\alpha)}{16h^2a^2}h^2\rho^2\right\}K_0(\iota h\rho) . \quad . \quad (29)$$

Determination of Inductance and Resistance.

The terms of the Fourier series involving ϕ do not contribute to the total current across a section of the wire. Let $Ee^{\iota pt}$ be the mean value across a section, of the line integral of impressed electric force per unit length perpendicular to the section, and parallel to the central curve. If $\varpi e^{\iota pt}$ be the total current, and (L, R) the effective self-induction and resistance per unit length of wire,

$$(L\iota p+R)\varpi=E. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Now at this point, the problem becomes identical in process with that of the straight wire, provided that the appropriate functions $A_0(k\rho)$ and $B_0(\iota h\rho)$ are used for the wire and dielectric respectively, in place of $J_0(k\rho)$ and $K_0(\iota h\rho)$ for the straight wire. Quoting therefore the result in the latter case, as developed in a former paper[*], namely, if r be the radius of the wire,

$$\frac{E}{\varpi}=\frac{2\mu\iota p}{kr}\left\{\frac{J_0(kr)}{J_0'(kr)}-\frac{k}{\mu h\iota}\frac{K_0(\iota hr)}{K_0'(\iota hr)}\right\}, \quad . \quad . \quad (31)$$

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[*] Phil. Mag. Feb. 1909, p. 259. The result requires a factor $-1/a$.

we find for the helix

$$\frac{\mathrm{E}}{\varpi}=\frac{2\mu\iota p}{kr}\left\{\frac{\mathrm{A}_0(kr)}{\mathrm{A}_0'(kr)}-\frac{k}{\mu h\iota}\,\frac{\mathrm{B}_0(\iota hr)}{\mathrm{B}_0'(\iota hr)}\right\}. \quad . \quad . \qquad (32)$$

Let

$$16\delta=\cos^2\alpha\,(5\sin^2\alpha-1), \quad . \quad . \quad . \quad . \qquad (33)$$

Then

$$\mathrm{A}_0(kr)=\left(1+\frac{\delta r^2}{a^2}\right)\mathrm{J}_0(kr)$$

$$\mathrm{A}_0'(kr)=\left(1+\frac{\delta r^2}{a^2}\right)\mathrm{J}_0'(kr)+\frac{2\delta r}{ka^2}\mathrm{J}_0(kr),$$

and to order $1/k^2a^2$

$$\frac{\mathrm{A}_0(kr)}{\mathrm{A}_0'(kr)}=\frac{\mathrm{J}_0(kr)}{\mathrm{J}_0'(kr)}\left\{1-\frac{2\delta r}{ka^2}\,\frac{\mathrm{J}_0(kr)}{\mathrm{J}_0'(kr)}\right\}.$$

Finally, if

$$\frac{\mathrm{J}_0(kr)}{kr\mathrm{J}_0'(kr)}=\mathrm{P}+\iota\mathrm{Q}, \quad \frac{\mathrm{K}_0(\iota hr)}{\iota hr\,\mathrm{K}_0'(\iota hr)}=\mathrm{P}'+\iota\mathrm{Q}', \quad . \qquad (34)$$

$$\frac{\mathrm{E}}{\varpi}=2\mu\iota p\,(\mathrm{P}+\iota\mathrm{Q})\left\{1-\frac{2\delta r^2}{a^2}(\mathrm{P}+\iota\mathrm{Q})\right\}$$

$$-2\iota p\,(\mathrm{P}'+\iota\mathrm{Q}')\left\{1-\frac{2\delta r^2}{a^2}(\mathrm{P}'+\iota\mathrm{Q}')\right\}. \qquad (35)$$

In determining the effect of the concentration of current upon the inductance and resistance, which in the former case is all that is sought, only the terms involving P and Q (and thus the resistivity) are needed. The effective resistance is altered from its steady current value of $\sigma/\pi r^2$ to the real part of E/ϖ or

$$\mathrm{R}=-2\mu p\mathrm{Q}+\frac{\mu pr^2}{4a^2}\cos^2\alpha\,(1-5\sin^2\alpha)\,\mathrm{PQ}, \quad . \qquad (36)$$

and the self-induction is

$$\mathrm{L}=2\mu\mathrm{P}-\frac{\mu r^2}{4a^2}\cos^2\alpha\,(1-5\sin^2\alpha)(\mathrm{P}^2-\mathrm{Q}^2). \quad . \quad . \qquad (37)$$

The values of P and Q are well known. With Lord Kelvin's definition of the functions ber x, bei x, where

$$x=\iota^{-\frac{1}{2}}kr=r\,(4\pi\mu p/\sigma)^{\frac{1}{2}},$$

$$\mathrm{J}_0(kr)=\mathrm{ber}\,x+\iota\,\mathrm{bei}\,x, \quad . \quad . \quad . \quad . \qquad (38)$$

and on reduction,

$$\left.\begin{aligned}-x\mathrm{Q}&=(\mathrm{ber}\,x\,\mathrm{bei}'x-\mathrm{bei}\,x\,\mathrm{ber}'x)/\{(\mathrm{ber}'x)^2+(\mathrm{bei}'x)^2\}\\ x\mathrm{P}&=(\mathrm{ber}\,x\,\mathrm{ber}'x+\mathrm{bei}\,x\,\mathrm{bei}'x)/\{(\mathrm{ber}'x)^2+(\mathrm{bei}'x)^2\}\end{aligned}\right\}. \qquad (39)$$

The values of these functions have been tabulated, yet as Russell has shown *, the tables need revision. But it is preferable, in place of a laborious tabulation, to use the asymptotic formulæ for these functions, which Russell develops in the same paper. We may distinguish two important cases, and it is convenient to recapitulate, at this stage, the meanings of the various symbols. The oscillation is assumed to be sine-shaped, and of frequency $p/2\pi$, μ and σ are the permeability and resistivity of the wire. The wire has a radius r of section, and is wound on a cylinder of radius a, such that r/a is small. The angle of the helix is α. Let n be the number of turns of the helix in a length z parallel to its axis. Since, in a helix, $z=a\theta\tan\alpha$, we have

$$n=\theta/2\pi=z/2\pi a\tan\alpha,$$

and therefore

$$\alpha=\tan^{-1}(z/2\pi na). \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Case of Small Frequency.

When $x=r(4\pi\mu p/\sigma)^{\frac{1}{2}}$ is less than 2, we may write

$$\left.\begin{aligned} x\mathrm{P} &= \frac{1}{4}x\left\{1-\frac{1}{24}\left(\frac{x}{2}\right)^4+\frac{13}{4320}\left(\frac{x}{2}\right)^8-\frac{647}{12^2 . 360 . 56}\left(\frac{x}{2}\right)^{12}\right\} \\ -x\mathrm{Q} &= \frac{2}{x}\left\{1+\frac{1}{12}\left(\frac{x}{2}\right)^4-\frac{1}{180}\left(\frac{x}{2}\right)^8+\frac{11}{12 . 28 . 30}\left(\frac{x}{2}\right)^{12}\right\} \end{aligned}\right\}, \quad (41)$$

and therefore

$$\mathrm{PQ}=-\frac{1}{2x^2}\left\{1+\frac{1}{24}\left(\frac{x}{2}\right)^4-\frac{13}{2160}\left(\frac{x}{2}\right)^8\right\}$$

$$\begin{aligned} \mathrm{P}^2-\mathrm{Q}^2= & \ \frac{1}{16}\left\{1-\frac{1}{12}\left(\frac{x}{2}\right)^4+\frac{67}{8640}\left(\frac{x}{2}\right)^8\right\} \\ & -\frac{4}{x^2}\left\{1+\frac{1}{6}\left(\frac{x}{2}\right)^4-\frac{1}{240}\left(\frac{x}{2}\right)^8\right\}, \end{aligned}$$

no higher order being necessary in the last two, as they will be multiplied by r^2/a^2.

The resistance becomes

$$\begin{aligned} \mathrm{R}= & \frac{\sigma}{\pi r^2}\left\{1+\frac{1}{12}\left(\frac{\pi\mu pr^2}{\sigma}\right)^2-\frac{1}{180}\left(\frac{\pi\mu pr^2}{\sigma}\right)^4+\frac{11}{12 . 28 . 30}\left(\frac{\pi\mu pr^2}{\sigma}\right)^6\right\} \\ & -\frac{\sigma}{32\pi a^2}\cos^2\alpha(1-5\sin^2\alpha)\left\{1+\frac{1}{24}\left(\frac{\pi\mu pr^2}{\sigma}\right)^2-\frac{13}{2160}\left(\frac{\pi\mu pr^2}{\sigma}\right)^4\right\}, \end{aligned} \quad (42)$$

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* *Loc. cit.*

and the self-induction is changed by the concentration of current to an amount

$$\mathrm{L}=\frac{1}{2}\mu\left\{1-\frac{1}{24}\left(\frac{\pi\mu pr^2}{\sigma}\right)^2+\frac{13}{4320}\left(\frac{\pi\mu pr^2}{\sigma}\right)^4-\frac{647}{12^2.360.56}\left(\frac{\pi\mu pr^2}{\sigma}\right)^6\right\}$$

$$-\frac{\mu r^2}{64a^2}\cos^2\alpha(1-5\sin^2\alpha)\left\{1-\frac{1}{12}\left(\frac{\pi\mu pr^2}{\sigma}\right)^2+\frac{67}{8640}\left(\frac{\pi\mu pr^2}{\sigma}\right)^4\right\}$$

$$+\frac{\sigma}{4\pi a^2p}\cos^2\alpha(1-5\sin^2\alpha)\left\{1+\frac{1}{6}\left(\frac{\pi\mu pr^2}{\sigma}\right)^2-\frac{1}{240}\left(\frac{\pi\mu pr^2}{\sigma}\right)^4\right\}. \quad (43)$$

In these results it is not possible to proceed to the limit of zero frequency, for it has been assumed that k^2a^2 is large. Although, therefore, the expression for R does not become $\sigma/\pi r^2$ when p is made zero, no error is involved on this account. Near $p=0$ another form of expression must be used.

The limitations of these results are determined by the considerations that kr shall be less than 2, and that r^3/a^3 and ka^{-4} shall be negligible. As an average practical case, we take a copper wire of radius 2 millimetres, and determine the requisite conditions for a three figure accuracy. In the first place, r^3/a^3 cannot be ignored, unless a is greater than $12r$ approximately. This condition is usually fulfilled. Even if it is not, the order of accuracy may extend to two figures for a much greater value of r/a, so that this limitation is not of great moment.

Secondly, kr is less than 2, so that in the present case, if f be the frequency, and $\sigma=1696$ C.G.S. units, approximately

$$\tfrac{1}{5}rf^{\frac{1}{2}}<2 \quad \text{or} \quad rf^{\frac{1}{2}}<10,$$

where r is in centimetres. The upper limit in frequency is therefore about 2500 per second.

Thirdly, $(ka)^{-4}$ may be neglected if $ka>10$, so that, to determine the lower limit, there is an inequality

$$af^{\frac{1}{2}}>50,$$

or if $a=12r$ in the most unfavourable case, f is about 400 per second. The formulæ, therefore, range between frequencies of 400 and 2500 for a wire of 2 millimetres radius, wound on an appropriate cylinder. In general the conditions for a three figure accuracy are

$$a > 12r, \quad af^{\frac{1}{2}} > 50, \quad rf^{\frac{1}{2}} < 10, \quad . \quad . \quad . \quad (44)$$

the radii being in centimetres.

Case of High Frequency.

We pass to the case of a higher frequency, whose lower limit is determined later. When

$$\lambda = x2^{\frac{1}{2}} = 2r\,(2\pi\mu p/\sigma)^{\frac{1}{2}}$$

is not less than 8, a four figure accuracy may be obtained by the use of the formulæ*

$$\left.\begin{aligned} \lambda P &= 1 - \frac{3}{4\lambda^2} - \frac{3}{2\lambda^3} \\ -\lambda Q &= 1 + \frac{1}{\lambda} + \frac{3}{4\lambda^2} \end{aligned}\right\}, \quad . \quad . \quad . \quad (45)$$

a small but convenient change in the notation having been introduced. Therefore

$$-\lambda^2 PQ = 1 + \frac{1}{\lambda} - \frac{9}{4\lambda^3}$$

$$\lambda^2(P^2 - Q^2) = -\left(\frac{2}{\lambda} + \frac{4}{\lambda^2} + \frac{9}{4\lambda^3}\right),$$

and finally

$$R = \left(\frac{\mu p\sigma}{2\pi r^2}\right)^{\frac{1}{2}}\left\{1 + \left(\frac{\sigma}{8\pi\mu p r^2}\right)^{\frac{1}{2}} + \frac{3}{4}\cdot\frac{\sigma}{8\pi\mu p r^2}\right\}$$
$$-\frac{\sigma}{32\pi a^2}\cos^2\alpha\,(1 - 5\sin^2\alpha)\left\{1 + \left(\frac{\sigma}{8\pi\mu p r^2}\right)^{\frac{1}{2}} - \frac{9}{4}\left(\frac{\sigma}{8\pi\mu p r^2}\right)^{\frac{3}{2}}\right\} \quad (46)$$

$$L = \left(\frac{\sigma\mu}{2\pi p r^2}\right)^{\frac{1}{2}}\left\{1 - \frac{3}{4}\,\frac{\sigma}{8\pi\mu p r^2} - \frac{3}{2}\left(\frac{\sigma}{8\pi\mu p r^2}\right)^{\frac{3}{2}}\right\}$$
$$+\frac{\sigma\cos^2\alpha\,(1 - 5\sin^2\alpha)}{32\pi p r a^2}\left(\frac{\sigma}{2\pi\mu p}\right)^{\frac{1}{2}}\left\{1 + 2\left(\frac{\sigma}{8\pi\mu p r^2}\right)^{\frac{1}{2}} + \frac{9}{4}\,\frac{\sigma}{8\pi\mu p r^2}\right\} \quad (47)$$

A three-figure accuracy in these results will be secured if the conditions

$$a > 12r, \quad rf^{\frac{1}{2}} > 70. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

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* Russell, *l. c.* p. 532.

are satisfied, in the case of copper wires. No further condition as regards a is here needed, as it is satisfied by virtue of the others. In the limiting case of very high frequency

$$R=\left(\frac{\mu p\sigma}{2\pi r^2}\right)^{\frac{1}{2}}-\frac{\sigma}{32\pi a^2}\cos^2\alpha\,(1-5\sin^2\alpha). \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$L=\left(\frac{\sigma\mu}{2\pi pr^2}\right)^{\frac{1}{2}}+\frac{\sigma\cos^2\alpha(1-5\sin^2\alpha)}{32\pi pra^2}\left(\frac{\sigma}{2\pi\mu p}\right)^{\frac{1}{2}}-\frac{3\sigma}{32\pi\mu pr^2}\left(\frac{\sigma\mu}{2\pi pr^2}\right)^{\frac{1}{2}}, (50)$$

and the change in resistance due to twisting tends to become independent of both the frequency and the radius of the wire, whereas the change in self-induction tends to vanish.

For the intermediate case, in which $rf^{\frac{1}{2}}$ lies between the limits of about 10 and 70, the formulæ (36, 37) must be used, provided that a satisfies the proper condition. These formulæ require the functions P and Q when x lies between 2 and 7.

In a paper recently presented to the Physical Society *, Mr. H. G. Savidge has given tables of these functions suitable for the case in question, together with graphs of the functions $-x$Q, xP of (39). The formulæ (36), (37) are therefore rendered sufficient for practical use.

ABSTRACT.

This paper deals with a determination of the effective resistance and inductance of a helical coil of great length, composed of thin wire, wound on a cylinder whose radius is large in comparison with that of the wire. The pitch of the winding is not small, so that the problem cannot be treated by the method of Cohen. The method employed depends upon the use of a type of "helical coordinates" defining the position of any point, and of the general theorem relating to orthogonal systems of coordinates. A solution is obtained for the internal and external forces, corresponding to a given impressed electromotive force, in the form of a Fourier series of which only the initial terms require calculation. The value of the effective current across any section is obtained, and thence the inductance and resistance. The results are of simple character, and are expressed in general in terms of the ber and bei functions of Lord Kelvin, whose results for a straight wire appear as a limiting case. Certain particular cases are worked out in detail and formulæ obtained in terms of elementary functions which are suitable for a very high or a low frequency.

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* *Suprà*, p. 105.

Their necessary limitations are also examined numerically. For a high frequency, it is found that the change of self-inductance due to twisting of the wire tends to vanish, and that the change of resistance tends towards a value independent of the frequency.

Discussion.

Dr. Russell stated that the Author's paper was most instructive and that he had done excellent pioneering mathematical work. This paper contained the first published attempt to get a solution of the very difficult problem of finding the effective resistance and inductance of a helical coil when traversed by high frequency currents. Previously only cylindrical current-sheets had been considered. He pointed out that, in the particular case when the wire was very thin, an approximate value of the inductance could be found by counting the linkages of the magnetic lines of force with the helical current. Even when the coil was of finite length this presented no great difficulty. In the Author's problem, however, the wire was of finite thickness and so the difficulties to be overcome were much greater. He considered that the Author's solutions would prove very helpful to other workers in electromagnetic theory.

Prof. C. H. Lees congratulated the Author, and remarked that both the functions occurring in Equation 39 had been tabulated by Mr. Savidge. Prof. Lees indicated the form of the graphs of these functions.

The Author, in reply to Prof. Lees, said the tabulation of the functions referred to was most valuable.

X. *Ductile Materials under Combined Stress.* By Walter A. Scoble, *A.R.C.Sc., B.Sc., Whitworth Scholar**.

[Plate III.]

Introduction.

The theory of combined stress, and the results recorded by earlier writers, were discussed in a previous communication †, and the literature of this subject has since been very fully reviewed, from an engineering standpoint, by Mr. L. B. Turner ‡ and Mr. C. A. Smith §. The most important experimental results obtained by other observers are given later.

The object of this paper is to consider further the results

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* Read November 26, 1909.

† Proc. Phys. Soc. London, vol. xx., and Phil. Mag. Dec. 1906.

‡ Engineering, Feb. 5, 1909. § Engineering, Aug. 20, 1909.

Fig. 1.

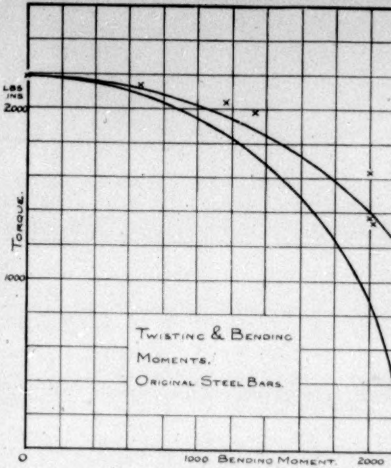
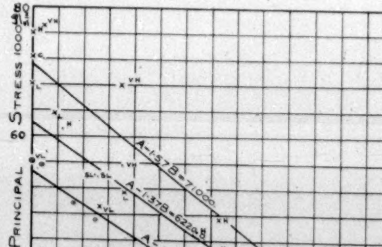
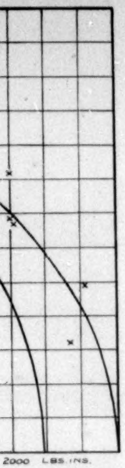


Fig. 4.





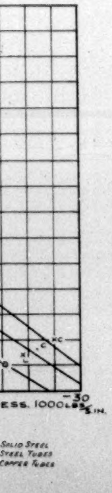


FIG. 2.

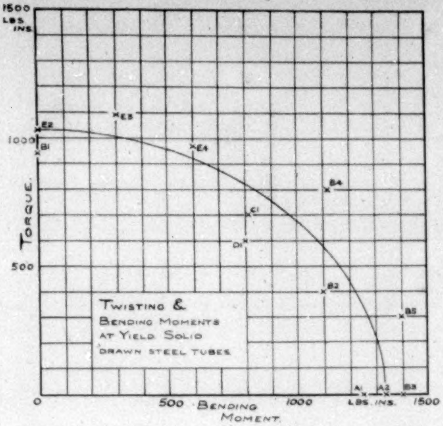


FIG. 3.

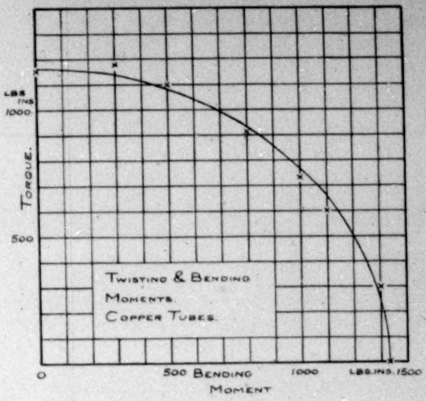


FIG. 5.

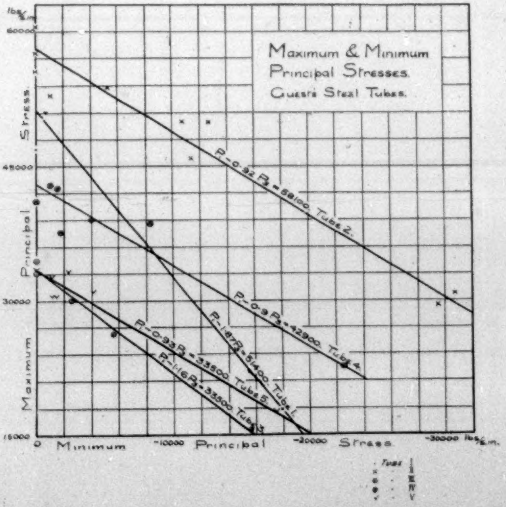
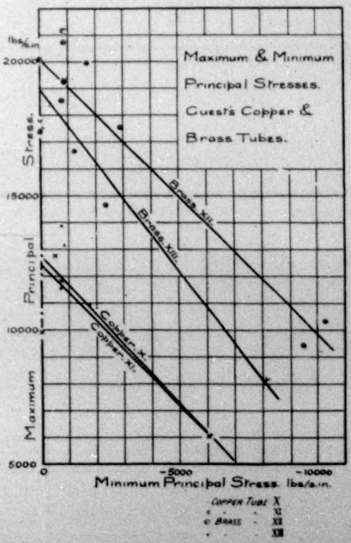


FIG. 6.



of the writer's earlier experiments, and to record the data which were obtained from some later tests made on tubes. The results are also compared with those obtained by other observers, who employed different methods and combinations of loading.

Further Consideration of the Earlier Tests.

Specimens and Apparatus.—The first test bars were of steel, ¾ inch diameter, and 36 inches long. The ends were of square section, ¾ inch side, to allow a torque to be applied to the bar. One squared end was held in a special clamp which prevented this end of the bar from rotating under the torque, but allowed it to take its natural slope as a supported beam under a bending load. The bar was also supported, at 30 inches from the centre of the clamp, on V rollers, which formed the other support under the bending load, and offered no resistance to the torsion of the bar. A bar was bent as a beam 30 inches long, supported at its ends, by a dead load directly applied midway between the supports; it was twisted by means of a wooden pulley which fitted on the other squared end of the bar. Two flexible wire ropes were attached to the pulley, and exerted a couple upon it because of the weights which they were made to carry.

Therefore it will be noticed that when a bar was twisted, it was subjected to a uniform torque all along its length, but the maximum bending moment acted on one section only of the bar. Within the elastic limit of the metal, the shear stress produced by the torque varied from zero at the axis of the bar, to a maximum all over the surface. Similarly the maximum stresses due to bending were a compression at the top of the bar, and a tension only at its lowest point; and these were confined to the mid-section, which was under the maximum bending moment. This system of loading represented the most common example of combined stress in engineering practice, which is a shaft subjected to combined bending and torsion, and it had the further advantage that the critical stresses were produced by comparatively small loads.

The bending deflexion was measured by means of a scale which rested on the bar, and was guided in a slide provided with a vernier. This simple arrangement was quite satisfactory on such a long beam. The torsion was measured by a pointer clamped to the bar, which moved over a fixed, finely-divided circle. The twist was also measured at three points by clamping mirrors to the bar; fixed telescopes and vertical scales were used in connexion with these mirrors.

The Criterion of Strength.

The theory of elasticity is based on Hooke's law, that strain is proportional to stress. Beyond the elastic limit of a material this law is no longer strictly true, and therefore the usual formulæ for calculating the stresses cannot be applied. Consequently it is clear that the elastic limit is the correct criterion of strength. But it is now the usual practice to adopt the yield-point. Guest * stated that Hooke's law holds to the yield-point, and he considered that the first deviation from proportionality of stress to strain was caused by local variations in the material, which altered the stress distribution, and caused local yielding. Thus he assumed that stress was proportional to strain for the main part of the material until the yield-point was reached, and therefore he selected the yield-point as the criterion of strength. Undoubtedly the yield-point of a material is less affected by special treatment than the elastic limit, and it is more easily determined, consequently other observers, following Guest, have adopted the yield-point.

Unfortunately opinions differ concerning the exact location of the yield-point, and for comparative purposes, in order to have a well-defined point, the writer also neglected the intermediate state between perfect elasticity and complete yield, and obtained the critical loads from the intersections of the continuations of the lines which represent these two limiting states on the stress-strain curves.

It is probable that Guest's assumption is partly true, because, with the loading adopted by the writer, with which the stresses are unequally distributed, the ratio of the strain to the stress is constant until it increases very rapidly just before the complete breakdown of the specimen. The slight deviation from Hooke's law, which is noticed between the elastic limit and the yield-point in a simple tension test, is not observed.

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* Proc. Phys. Soc. London, vol. xvii. Sept. 1900.

It therefore appears that for this kind of test the elastic limit and the yield-point practically coincide, and therefore the stresses are now taken, in all these tests, from the point where the strain ceases to be proportional to the stress.

The Quantities Tabulated.

The strains have not been tabulated because the maximum strain theory is not supported by engineers in this country. The formulæ relating to combined stress which are given in the text-books are based on the maximum stress theory. Recently, however, the shearing stress theory, or the stress difference theory as elasticians prefer to name it, has rapidly gained favour with engineers. It is, therefore, clear that the maximum stress and the maximum shearing stress are most important from a practical standpoint, but the maximum strain hypothesis is indirectly considered later, when the deviations from the shear-stress law are discussed.

Calculation of the Stresses.

The maximum tensile stress, p, in the material due to bending is calculated from the formula

$$\frac{p}{y} = \frac{\mathrm{M}}{\mathrm{I}},$$

in which

M is the maximum bending moment ;

I is the moment of inertia of the area of the section about its neutral axis, in this case a diameter ;

y is the greatest distance of a point in the section from the neutral line, and equals half the diameter of the section.

The maximum shear stress, S, caused by the torque, is calculated from the formula

$$\mathrm{T} = \frac{\pi}{16}\mathrm{SD}^3,$$

in which

T is the torque ;
D is the diameter of the bar.

For the tubes employed in the later tests the formula becomes

$$T = \frac{\pi}{16} S \frac{D^4 - d^4}{D},$$

in which

D is the external diameter of a tube ;
d is its internal diameter.

Having found p and S, the maximum and minimum principal stresses are represented by the expressions

$$\frac{p}{2} \pm \sqrt{\frac{p^2}{4} + S^2},$$

one of which is positive, and the other is negative. The third principal stress is zero, because in the case of solid bars the stress due to the shearing force on the section is zero where the bending stress is a maximum. Since the tubes are bent by couples, there is no direct shearing force on a cross-section.

The stress difference is the difference between the maximum and minimum principal stresses, and therefore equals

$$2\sqrt{\frac{p^2}{4} + S^2}.$$

It is twice the maximum shearing stress.

The maximum strain is given by

$$\frac{P_1 - \eta(P_2 + P_3)}{E},$$

in which

P_1 is the maximum principal stress ;
P_2 and P_3 are the other principal stresses ;
η is Poisson's Ratio ;
E is Young's Modulus of Elasticity.

In these tests there are stresses in one plane only, and the expression becomes

$$\frac{P_1-\eta P_3}{E}.$$

If the maximum strain be constant, then $P_1-\eta P_3$ must be constant, so that this hypothesis comes between the maximum stress law, that P_1 is constant, and the stress difference or shear-stress theory, that P_1-P_3 is constant, but it is nearer the former.

The Results from Steel Bars.

TABLE A.—Original Tests of Solid Steel Bars.

Number of Bar.	Bending Moment. lbs. ins.	Twisting Moment. lbs. ins.	Tensile Stress due to Bending. lbs./s. in.	Shear Stress due to Torque. lbs./s. in.	Maximum Principal Stress. lbs./s. in.	Minimum Principal Stress. lbs./s. in.	Stress Difference =2co Maximum Shear Stress.
III. ...	0	2190	0	26600	26600	−26600	53200
IV. ...	667·5	2130	16220	25880	35260	−19040	54300
XI. ...	1160	2033	28200	24700	42520	−14320	56840
V. ...	1331	1985	32350	24100	45210	−12860	58070
XII. ...	2000	1360	48600	16520	53700	− 5100	58800
VI. ...	2000	1630	48600	19800	55650	− 7050	62700
IX. ...	2020	1335	49000	16220	53900	− 4900	58800
VIII. ...	2320	645	56300	7840	57350	− 1050	58400
VII. ...	2420	980	58750	11900	61075	− 2325	63400

Only one test was made on each bar. The bending and twisting moments at yield are plotted in fig. 1 (Pl. I.). If the maximum principal stress were constant at the yield-point, the twisting moment under pure torque would be double the bending moment required to cause the bar to yield. If the maximum shearing stress were constant, the plotted points would lie on a circle described with the origin as centre. Fig. 1 confirms the writer's original conclusion that the maximum shear stress is approximately constant at failure, but that an ellipse lies between the points better than a circle, and that the bending moment is greater than the torque.

The maximum stress varies between 26,600 and 61,075 lbs./s. in., it certainly is not constant. The stress difference is not exactly constant; it varies from 53,200 to 63,400 lbs./s. in., but it increases steadily with the bending moment, consequently the deviation from the stress difference, or shear stress law, is still further away from a constant maximum stress, so that the maximum strain also varies considerably.

The Apparatus for the Tests on Tubes.

The tests on tubes were made in a machine which is part of the equipment of the engineering laboratory at the City and Guilds Technical College, Finsbury. The machine was designed by Prof. E. G. Coker, and it has been recently described*. A tube to be tested was sweated, and when it was necessary pinned, on two steel holders, or mandrils, each of which was provided with two keyways. One holder fitted into a wormwheel so that this end of the tube could be twisted, but the wormwheel casing was pivotted on roller bearings so that there was no resistance to bending. A lengthening piece was fitted on the holder to carry an overhung load, which produced a bending moment on the specimen. The other holder was held in a special fitting which was supported on a spindle of small diameter, so that the resistance to torsion was negligible. The fitting was also arranged to carry an overhanging load at this end, and an arm projected at right angles to the specimen to support the load which measured the torque. This latter arm was always kept horizontal by turning the wormwheel as the twisting load was increased. The loading in this machine was also by dead weights. The torque was produced by the twisting load and the extra supporting force which it required at its support. The torque was uniform along the tube. The bending moment was caused by a couple at each end, composed of the supporting force and the overhanging load; it was constant along the specimen. There was no shearing force on a section due to the bending loads.

The twist of a tube was measured by Prof. Coker's torsionmeter, and the bending deflexion by an adaptation of the Ayrton-Perry twisted strip. The latter apparatus was designed for use during some bending tests in which the deflexion was extremely small, and it was therefore very sensitive for the purpose of the present tests. It was necessary to measure the strains separately, because the writer has shown that when a ductile material is under combined loading, it does not always yield first in the way which is indicated by the increasing load. The first yield is probably determined by the loading which produces the greatest shearing stress.

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* Proc. Phys. Soc. London, vol. xxi., and Phil. Mag. April 1909.

The Tests on Steel Tubes.

Solid drawn steel tubes were tested, and the yield during each test was kept very small so that several experiments could be made with each tube. All these specimens were cut from the same length of tube. The results are collected below.

TABLE B.—Tests of Solid Drawn Steel Tubes.
Internal diameter 0·818 in. External diameter 0·885 in.

Test.	Bending Moment. lbs. ins.	Twisting Moment. lbs. ins.	Tensile Stress due to Bending. lbs./s. in.	Shear Stress due to Torque. lbs./s. in.	Maximum Principal Stress. lbs./s. in.	Minimum Principal Stress. lbs./s. in.	Stress Difference = 2ce Maximum Shear Stress.
B 1 ……	0	940	0	25600	25600	−25600	51200
E 2 ……	0	1030	0	28050	28050	−28050	56100
E 3 ……	300	1090	16330	29700	38970	−22630	61600
E 4 ……	600	970	32700	26400	47400	−14700	62100
D 1 ……	800	600	43550	16330	48980	− 5420	54400
C 1 ……	810	700	44100	19100	51250	− 7150	58400
B 2 ……	1100	400	59900	10900	61770	− 1870	63640
B 4 ……	1120	800	61000	21800	68000	− 7000	75000
A 1 ……	1250	0	68100	0	68100	0	68100
A 2 ……	1330	0	72400	0	72400	0	72400
B 3 ……	1400	0	76200	0	76200	0	76200
B 5 ……	1400	300	76200	8165	77000	− 800	77800

The maximum stress has 25,600 and 77,000 lbs./s. inch for its extreme values, and the stress difference varies from 51,200 to 77,800 lbs./s. in. The bending and twisting moments are plotted in fig. 2, in which the numbers of the tests are shown against the points. The letter refers to the tube, and the number indicates the order in which the test was made on that specimen.

The figure is interesting because it shows very clearly the effect of repeated loading. An ellipse is drawn to lie evenly between the points. Specimens cut from the same length of material are not exactly alike, and an error is possible in locating the yield-point; but after allowance is made for these facts, it is evident that the yield-point was raised by a previous test. Nevertheless, the results clearly indicate that the maximum shear stress is more nearly constant than the principal stress, and that the bending moment is again greater than the torque, so that the shearing stress and the stress difference increase with the bending moment.

The Tests on Copper Tubes.

Tests were also made on solid drawn copper tubes, of 0·79 inch internal and 0·881 inch external diameter. The specimens were given a set to correct for their defective elasticity, and to allow several tests to be made on each tube. As in the case of the steel tubes, the yield during each test was kept small. The data obtained from these tests are tabulated below.

TABLE C.—Tests of Solid Drawn Copper Tubes.
External diameter 0·8812 inch. Internal diameter 0·790 inch.

Test.	Bending Moment. lbs. ins.	Twisting Moment. lbs. ins.	Tension due to Bending. lbs./s. in.	Shear Stress due to Twisting. lbs./s. in.	Maximum Principal Stress. lbs./s. in.	Minimum Principal Stress. lbs./s. in.	Stress Difference =2ce Maximum Shear Stress.
A 2	0	1150	0	24200	24200	−24200	48400
A 3	300	1180	12620	24800	31910	−19290	51200
C 3	500	1100	21040	23150	35920	−14880	50800
C 2	800	910	33660	19140	42330	− 8670	51000
C 4	1000	730	42080	15360	47080	− 5000	52080
B 3	1100	600	46300	12620	49510	− 3210	52720
B 2	1300	300	54700	6310	55450	− 750	56200
B 4	1330	0	56000	0	56000	0	56000

Here again the stress difference is approximately constant, and the deviation from this law is opposed to a constant maximum stress, because the stress difference increases steadily with the bending moment. The bending moment and the torque are plotted in fig. 3.

Deviations from the Shear-Stress Law.

The stress difference is given in the tables, because it is the difference between the maximum and minimum principal stresses, which it follows. It is now more convenient to deal with the maximum shear stress, which is half the stress difference. The stress difference and maximum shear-stress laws are therefore practically alike, but the former does not indicate the existence of the shearing stress which appears to cause the actual fracture of a ductile material. A ductile material behaves like a viscous fluid after the yield-point, so that it would be expected that the flow is caused by a shearing stress. The stress difference theory indicates a result, but it does not indicate the behaviour of the material. Moreover, the shearing is the stress considered by engineers.

By referring to the tables it will be seen that the maximum principal stress increases tremendously as the bending moment increases, whereas the maximum shear stress is approximately constant. If the maximum stress were constant, the maximum shear stress would decrease with increasing bending moment; but it is found that the shear stress increases, so that the deviation from the shear-stress law is opposed to the maximum stress theory, and it disproves the maximum strain hypothesis, which is in effect an intermediate law, as has been already explained. The results are similar for each series of tests.

Even before there were any reliable experimental results available, certain elasticians, influenced possibly by the forms of fractures, were convinced that a material failed by shearing. But further, there was usually a stress perpendicular to the plane of greatest shear, and it was believed that this force across the plane affected the value of the shear stress at the elastic failure of the material. The shear stress tended to make the opposite surfaces at a section slide upon each other, and therefore it appeared that a compression across the section would introduce a resistance to the shear analogous to friction. Since a material tended to flow or fracture along an uneven section, so that the surfaces fitted into each other to a certain extent, it was conceivable that a tension across a section assisted the shearing stress, and lowered its value at the breakdown.

Many of the experimental results available were examined in the former paper to determine whether the friction hypothesis would explain the deviation from the shear-stress law. The final conclusion was that :—" It must be concluded that the maximum shear stress determines when yield takes place, but this will vary slightly on account of the difference in the shearing resistance in various directions, and any idea of a force analogous to friction must be abandoned."

But the bending moment appears to be always greater than the torque, and it was suggested that when a shaft was under a torque, T, and a bending moment, M, the equivalent torque, T_e, should be calculated from the equation

$$T_e^2 = T^2 + \left(\frac{2f_s}{f_t}\right)^2 M^2,$$

in which f_s is the shear stress, and f_t is the tensile stress in the material at yield under simple loading. This equation represents an ellipse which replaces the circle of the shear-stress law, and thus allows for the limiting bending moment being greater than the torque. The present experimental results justify the above suggestion.

The theories of failure under combined stress can be very simply expressed in terms of the three principal stresses, P_1, P_2, and P_3, of which P_1 is the greatest and P_3 is the least. The maximum stress hypothesis assumes that $P_1 = \text{constant}$. If the maximum strain theory be correct, then, for stresses in two directions only, $P_1 - \eta P_3 = \text{constant}$. The stress difference or shear-stress law states that $P_1 - P_3 = \text{constant}$. Since all these equations are particular forms of $P_1 + mP_3 = \text{constant}$, it appears to be desirable to examine the results to find whether the stress distribution at yield may be expressed in a similar form.

The maximum and minimum principal stresses for these tests are plotted in fig. 4. The points for the copper tubes lie very close to a straight line because the original results were exceptionally good. The other points would have been close to the straight lines if corrected values had been taken, but, if the remarks against the points are considered, the lines represent the relations between the principal stresses fairly well. The letters against the points in fig. 4 indicate the positions of the corresponding points in the former diagrams when compared to the mean curves. The letter indicates the error in the original result from all causes, the experimental error, difference in material, and in the case of the steel tubes, due to the effect of repeated loading. On the diagram, fig. 4, A is the maximum and B is the minimum principal stress. Using the original notation, the equations which express the relations between the principal stresses are

$$P_1 - 1{\cdot}57\,P_3 = 71000,$$
$$P_1 - 1{\cdot}37\,P_3 = 62200,$$
$$P_1 - 1{\cdot}26\,P_3 = 54500.$$

The constant varies because it indicates the strength of the material, and is the tensile stress at yield. The values for "*m*" are 1·57, 1·37, and 1·26, but the materials are different.

It is now evident that the shear-stress law is most nearly correct of those stated above, and it is intended to be applied to all ductile materials. Although the deviation from the shear-stress law is sometimes considerable (under combined bending and twisting), if an equation be adopted to express the conditions at yield more closely, it must contain a constant which depends upon the material considered. This is a very serious complication, and engineers will probably prefer to assume a constant shear stress at failure, although the results obtained will be slightly inaccurate. Also combined bending and torsion is not the only example of combined loading, and it is desirable for practical purposes to adopt one rule to apply to all cases, if that be possible. It is therefore proposed to further consider this matter by examining the results obtained by other observers, who employed different methods and combinations of loading.

Guest called these principal stresses P_1, P_2, and P_3, and each represented the stress in a particular general direction, irrespective of its relative magnitude. The notation has

Results obtained by other Observers.

GUEST. Proc. Phys. Soc. London, Sept. 1900, vol. xvii.

Loading by Tension, Torsion, and Internal Pressure.

STEEL TUBE I.

P_A.	P_B.	P_C.
17900	−17900	0
50300	9000	−430
51050	13500	−640
51300	14800	−700
46800	0	0
17900	−17900	0

COPPER TUBE X.

P_A.	P_B.	P_C.
6070	−6070	0
10630	0	0
10950	−1755	0
11500	10240	−660
10830	13160	−850
12050	0	0
12750	0	0
13835	3355	−740

STEEL TUBE II.

P_A.	P_B.	P_C.
30800	−30800	0
60500+	0	0
29500	−29500	0
45800	−11350	0
55500	0	0
57800	0	0
53900	− 5400	0
29500	−29500	0
49900	−10800	0
49800	−12600	0
52900	22000	−1050

COPPER TUBE XI.

P_A.	P_B.	P_C.
5925	11850	−750
9925	0	0
12730	7740	−500
6070	−6070	0

STEEL TUBE III.

P_A.	P_B.	P_C.
15650	−15650	0
34400	0	0
15650	−15650	0
26250	− 5650	0
30050	− 2550	0
33000	0	0

BRASS TUBE XII.

P_A.	P_B.	P_C.
9450	−9450	0
17400	0	0
16650	−1215	0
14685	−2315	0
20770	14500	−850
19260	15360	−900
20100	0	0
10350	−10350	0
17610	− 2940	0
19980	16800	−985
21150	14500	−850
18520	1040	−750

STEEL TUBE IV.

P_A.	P_B.	P_C.
22500	−22500	0
41200	0	0
38650	−8350	0
42900	24000	−1050
37700	−1700	−1050
42700	33800	−1500
39000	−4000	0
41200	0	0
22500	−22500	0

BRASS TUBE XIII.

P_A.	P_B.	P_C.
19250	15340	−900
17750	0	0
8100	−8100	0

STEEL TUBE V.

P_A.	P_B.	P_C.
33500	0	0
17500	−17500	0
30900	− 4100	0
33100	− 2300	0
32600	24000	−1050
30500	5500	−1050
30350	29800	−1350

been changed to P_A, P_B, and P_C, because P_1, P_2, and P_3 have been reserved to represent the principal stresses in descending order of magnitude.

C. A. Smith. 'Engineering,' Aug. 20, 1909.

Table A.—Solid Steel. Combined Compression and Torsion.		Table C.—Solid Mild Steel. Tension, Compression, and Torsion.		Table D.—Hollow Steel. Compression and Torsion.	
Max. Principal Stress. lbs./s. in.	Max. Shear Stress. lbs./s. in.	Max. Principal Stress. lbs./s. in.	Max. Shear Stress. lbs./s. in.	Max. Principal Stress. lbs./s. in.	Max. Shear Stress. lbs./s. in.
19800	19500	−20170	19450	24900	23700
19800	19500	−20560	20420	47600	23800
26100	20000	−31810	21000		
29500	20400	−31570	20700		
32300	20400	−35620	17810		
34900	19900	−38180	19190		
36500	18900	−20750	20610		
		27950	20320		
		30140	19710		
		26370	20500		
		29050	20420		
		32300	21300		
		36420	18210		
		20560	20420		
		−19150	19050		
		−26210	20320		

The maximum and minimum stresses for Guest's steel tubes are plotted in fig. 5, and for his brass and copper tubes in fig. 6 (Pl. III.). Mean lines are drawn through the points, and their equations are :—

Guest's Steel Tube	I.	$P_1 - 1{\cdot}87\,P_3 = 51400$.
" " "	II.	$P_1 - 0{\cdot}92\,P_3 = 58100$.
" " "	III.	$P_1 - 1{\cdot}16\,P_3 = 33500$.
" " "	IV.	$P_1 - 0{\cdot}9\,P_3 = 42900$.
" " "	V.	$P_1 - 0{\cdot}93\,P_3 = 33500$.
Guest's Copper Tube	X.	$P_1 - 1{\cdot}09\,P_3 = 12700$.
" " "	XI.	$P_1 - 1{\cdot}04\,P_3 = 12400$.
Guest's Brass Tube	XII.	$P_1 - 1{\cdot}03\,P_3 = 20100$.
" " "	XIII.	$P_1 - 1{\cdot}34\,P_3 = 19000$.

Mr. Smith's results are not shown plotted here, but they are found to give,—

Smith's Solid Steel. Series SC. Table A. $P_1-1{\cdot}09\ P_3=41000$.
„ „ „ Series AD. Table C. $P_1-0{\cdot}775\ P_3=36300$.
„ Hollow Steel. Series SB. Table D. $P_1-1{\cdot}01\ P_3=47600$.

It is evident from the diagrams that the points are not always very close to the straight lines, so that it is difficult to assign an exact value to "*m*" for a given series of tests. But when average values are taken for "*m*," they are found to differ considerably for different tests, even when these are made under very similar conditions. For steel tubes, Guest's "*m*" varies from about 0·9 to 1·9. The copper tubes give consistent results, but one brass tube gives "*m*" equal to 1·03, and the other has "*m*" equal to 1·34; the points are very irregular. It is evidently not necessary to give a more elaborate method for finding "*m*," since it varies so much. Mr. Smith's values for steel are 1·09, 1·01, and 0·775.

An examination of Guest's results shows that the third principal stress has no appreciable effect on the other stresses at failure.

Conclusion.—In most cases the deviations from the shear-stress law are opposed to a constant maximum stress, and this is always so with bending. But it is probable that the value of "*m*" varies somewhat for ductile materials, because there are degrees of ductility. The writer has shown that cast-iron behaves quite differently to a ductile material *, but it does not conform to any exact law. He hopes shortly to publish results which prove that a strictly brittle material behaves differently to cast-iron and ductile materials. It is therefore not surprising that the results from ductile materials vary somewhat, and it is desirable that a stress-strain diagram for a tension test of the material should be considered in order to estimate its ductility. It is possible that the behaviour of all materials might be expressed in one form, P_1+mP_3=constant, in which m depends on the degree of ductility of the material. But in the present state of our knowledge it may be fairly claimed that the shear-stress or stress-difference law expresses the average behaviour of ductile materials under combined stresses, and that the maximum stress and maximum strain laws are not true for ductile materials.

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* Proc. Phys. Soc. vol xx.

ABSTRACT.

The Author further considers the results from some earlier tests made on mild steel bars, 3/4 inch diameter, and 30 inches effective length, under combined bending and torsion. It is pointed out that the yield-point is usually selected as the criterion of strength, because it is more easily determined than the elastic limit, it is less affected by special treatment of the material, and it is assumed that the failure of Hooke's Law between the elastic limit and the yield-point is due to local yielding. The elastic limit is the correct point, and is used throughout because the intermediate state mentioned above does not appear in bending. The results of tests on steel and copper tubes under combined bending and torsion are also given. All the results indicate that the maximum stress and maximum strain laws do not apply to ductile materials. The stress difference or shear stress law is approximately true, but there is, in each case, a deviation from the law which is opposed to the other theories mentioned. The deviations from the shear stress law are considered. In an earlier paper the Author suggested a formula for combined bending and torsion which allows for the fact that the bending moment is always greater than the torque. The internal friction hypothesis was also shown to be untenable. The three laws are now expressed in terms of the principal stresses, P_1, P_2, and P_3, of which P_1 is the greatest, and P_3 is the least. Guest's experiments proved that P_2 does not appreciably affect the failure of a material. The maximum stress law states that P_1=constant; the maximum strain theory that $P_1 - \eta P_3$ = constant, in which η is Poisson's Ratio; and the stress difference or shear stress hypothesis is expressed in the form $P_1 - P_3$ = constant. In the general equation $P_1 - mP_3$ = constant, the value of "m" indicates which law is most nearly true for the material. The Author's tests appear to be the only experiments in which bending was adopted, and for these the values of "m" are 1·57, 1·37, 1·26. The figures apply to different materials, but are all greater than unity.

The results obtained by other observers, who employed different methods and combinations of loading, are also examined. For Guest's steel tubes, "m" varies from 0·9 to 1·87; his copper tubes give 1·04 and 1·09. For brass tubes the values are 1·03 and 1·34. Smith's tests on steel, three series, show "m" to vary from 0·775 to 1·09.

The shear stress law appears to state the average behaviour of ductile materials, but there are considerable deviations from the law, which are usually opposed to the other theories. Other tests by the Author indicate

that brittle materials obey the maximum stress law, and it is therefore suggested that the value of "*m*" depends chiefly on the degree of ductility of the material considered, and to a lesser extent on the system of loading.

DISCUSSION.

Prof. **COKER** congratulated the Author on being able to present so many actual tests. Although many experiments had been carried out during the past 10 years there was still difficulty in stating the law which was followed when a ductile material was subjected to stress. An initial difficulty was to determine whether the elastic limit or the yield-point should be selected as the criterion of strength. Guest had shown that for tension both points yielded similar results, but for combined stress the results were very different. He thought the elastic limit should be selected. With reference to the repeated loading of the tubes he thought more consistent results would have been obtained if a new tube had been used for each test. Although the results obtained threw light on the various theories they did not favour any one of them.

Mr. C. A. **SMITH** expressed his interest in the paper and said he did not think much further progress would result from experiments on tubes. He enumerated the disadvantages connected with the use of tubes and said he thought the case for the use of solid rods was proved. He did not think copper specimens were suitable for tests, as the same system of loading did not always give the same result. Mr. Smith pointed out that no reference had been made to non-axial loading. If the load was applied acentrically errors were introduced which should be allowed for. He thought it was advisable to work with as simple a formula as possible ($P_1 - P_3 =$ cons.) and to introduce a suitable constant for each different material.

Mr. **SCOBLE**, in reply to Prof. Coker, said that Guest used the yield-point as the criterion of strength. In his paper he (the Author) had referred his stresses to the elastic limit, *i. e.* the point at which the strain ceased to be proportional to the stress. He had used solid bars of ¾ inch thickness because engineers required results on specimens as large as possible. He appreciated the advantage of using a simple law, but the results of his experiments did not follow any simple law.

XI. *The Recoil of Radium C from Radium B.*

By WALTER MAKOWER, *M.A., D.Sc., Assistant Lecturer and Demonstrator, and* SIDNEY RUSS, *D.Sc., Demonstrator in Physics, in the University of Manchester* [*].

INTRODUCTION.

IT has been shown in a previous paper [†], that during a radioactive transformation involving the expulsion of an α particle, the residue of the atom from which the α particle has been expelled recoils in an opposite direction to that in which the α particle is emitted and can travel a considerable distance through a gas if the pressure is sufficiently low. A similar effect was also demonstrable in the case of the transformation of radium B into radium C, although this change is supposed to be accompanied by the expulsion of only β rays, no α rays having ever been detected. (It has recently been shown by Hahn and Meitner that radium C is complex; but it is only the product whose period is 19 minutes with which we are concerned in what follows [‡].)

Now it was thought that this transformation was worthy of further study, since it is possible to investigate it by itself without any complications arising from other radioactive processes taking place simultaneously. If a plate is exposed as the negative electrode in an electric field to radium emanation for a long time, radium A. radium B, and radium C will be found on the plate after removal from the emanation; but if the plate is then left for half or three-quarters of an hour, the radium A on the plate will have diminished to a small fraction of its initial value, and we are left with only radium B and radium C, which are being transformed respectively to radium C and radium D. A disk which is suspended above such a plate therefore recieves radium C and radium D projected on it as a result of the recoil during the transformation of radium B and radium C. Now, since the time period of radium D is exceedingly long, the activity of the disk due to the presence of this substance after exposure to the active plate is inappreciable, and we should therefore be in a position to study the disintegration of radium B by itself.

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[*] Read November 26, 1909.

[†] Russ & Makower, Proc. Roy. Soc. A. vol. lxxxii. 1909.

[‡] Hahn & Meitner, *Phys. Zeitschr.* x. no. 20, pp. 697–703 (1909).

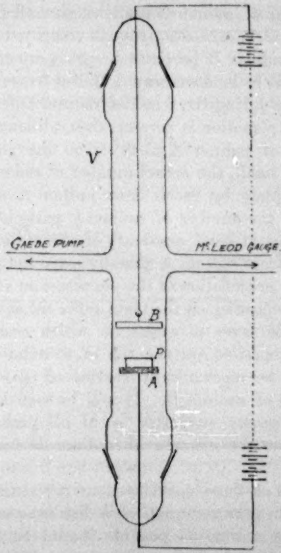
The phenomena were, however, soon found to be more complicated than had been anticipated, and it was found possible only under certain conditions to study the recoil of radium C from radium B without the interference of secondary disturbances.

Before discussing these conditions it will be necessary to briefly describe the apparatus used and the method of conducting an experiment, which were essentially similar to those employed in the experiments quoted above.

The glass vessel V (fig. 1), of 1·9 cms. diameter, was provided with stoppers at both ends. A brass support A was fixed as shown in the diagram, and an active plate P which was to serve as the radiating surface could be placed upon it. From the other stopper a disk was suspended by a wire and exposed to the radiation from the plate mounted on A. After an exposure of the required time with the air in V at the desired pressure, the disk B could be removed and tested for its activity in the usual manner, either by means of a quadrant electrometer or a sensitive α ray electroscope.

The method of experimenting was as follows:—A platinum plate, mounted on an iron disk to give it rigidity, was exposed in an electric field to radium emanation over mercury for at least three hours to allow the maximum amount of active deposit to be collected on the platinum surface. After removal from the emanation the iron-platinum plate was allowed to stand for a time varying in different experiments from half an hour to three-quarters of an hour before mounting on the support A (fig. 1). This interval was used to get rid of the emanation dragged out, from the vessel containing the emanation, by the disk and adhering to it. It was no easy matter to do this satisfactorily, but after a number of trials the best plan was found to be to put the plate in an evacuated tube kept at about 360° C. by an electrically heated furnace. Air was occasionally admitted and pumped out again to remove the emanation as it was given off by the plate. In this way, the adhering emanation was almost completely removed from the plate without appreciable loss of radium B by volatilization. The quantity of active deposit on the plate after this treatment was then tested by a γ ray electroscope in the usual manner by comparison with a standard quantity of radium.

Fig. 1.



Conditions for obtaining Pure Radium C by Recoil.

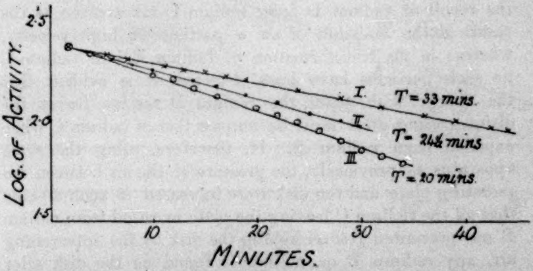
Attention has already been drawn to the fact that it is only under special conditions that the recoil of radium C from radium B can be studied by itself without the interference of disturbing causes. It was soon found that if the disk B (fig. 1) were exposed to the active plate P in a vacuum, there was in general radiated to the disk, not only radium C, as was to be expected, but also a certain amount of radium B.

Now, since in all the experiments to be described in this paper, a long interval of time was allowed to elapse between withdrawing the active plate from the emanation and mounting it for an experiment, the proportion of radium A left on the plate must have been very small. It was therefore somewhat surprising that any radium B should reach the disk, and the facts seem capable of only two explanations. Either the proportion of radium C particles shot off from an active plate by recoil is exceedingly small compared with the total number of radium B particles breaking up on the plate, or the radium B is in some way detached from the plate and carried away during the recoil of radium D from radium C. If the first explanation is correct, then, although the ratio of the number of radium A particles on the plate to that of radium B is small, the actual number of radium B particles leaving the plate by recoil from radium A might be comparable with the number of radium C particles. It will be seen later that at least one-tenth of the radium B particles which recoil from radium A actually leave the plate.

The second explanation of the phenomenon seems probable if there is deposited on the plate sufficient active deposit to form a double layer of atoms, in which case a radium B particle, if deposited on the top of a radium C particle, might easily be mechanically carried off the plate during the formation of radium D. It will be seen later that each of the two causes suggested is in all probability partly responsible for the ejection of radium B from the active plate.

On account of these considerations it was thought that in order to obtain pure radium C on a disk exposed to an active plate, as long a time as possible should be allowed after removing the plate from the emanation to allow the radium A on it to decay to an inappreciable quantity. Moreover, it seemed desirable to use the smallest quantities of emanation that would permit of the subsequent measurements being made with sufficient accuracy. The following series of experiments shows how a condition can thus be reached in which the active plate radiates practically pure radium C to a disk exposed to it in a high vacuum. The plate to be rendered active was exposed for some hours to a considerable quantity of emanation, after which it was withdrawn and heated in a vacuum for 27 minutes to remove any adhering emanation. The quantity of radium C on the plate was then measured on a γ ray electroscope, and found to be equal to the amount in equilibrium with 8·05 mgms. of radium bromide. Six minutes later it was mounted on the support A (fig. 1), and a disk suspended a few millimetres above it to receive the radiant active matter emitted by it. After an exposure of 20 minutes the disk was removed, and the rate of decay of the activity collected on it measured by a quadrant electrometer. The results obtained are shown in fig. 2, curve I. (p. 104). It will be seen that the activity of the disk fell to half value in about 33 minutes, indicating that there were both radium B and radium C on it. After 80 minutes from the time at which the active plate was removed from the emanation, a second disk was exposed to the radiating plate, and a similar series of observations made with it after an exposure of 20 minutes. The results are given in curve II. and show that even after this time the disk had received some radium B as well as radium C, for the activity took $24\frac{1}{2}$ minutes to fall to half its initial value, although the radium A on the plate must have fallen to a hundred millionth of its initial value by the time this exposure was begun. A third series of observations was therefore made with another disk beginning 184 minutes after the removal of the plate from the emanation. By this time the quantity

Fig. 2.

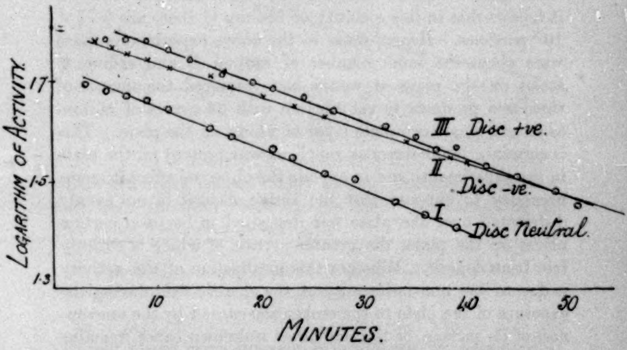


of active deposit on the plate had become so small, that on testing the disk its activity was found to fall to half value in 20 minutes, indicating that the matter radiated to the disk now consisted of practically pure radium C (see curve III.).

The experiments just described indicate that even when there was practically no radium A on the plate, a certain amount of radium B reached the disk suspended above it *in vacuo*; and, as we have seen, this phenomenon can be explained if we suppose that radium B is mechanically removed from the plate during the recoil of radium D from radium C. Since, in addition to the interest of the phenomenon itself, this action is liable to produce disturbances when studying the recoil of radium C from radium B, the matter was investigated in greater detail. Now, since the recoil of radium D from radium C takes place as the result of the emission of an α particle of high velocity, whereas in the transformation of radium B into radium C no such particles have been detected, it is evident that the energy with which the radium D residue leaves the disintegrating atom must far surpass that of radium C when expelled from radium B. If, therefore, using the same apparatus as previously, the pressure of the air between the radiating plate and the disk were increased to such a value that all the radium C leaving the plate by recoil from radium B was prevented from reaching the disk by the intervening air, any radium B or radium C found on the disk after exposure could only have reached it by secondary mechanical projection during the more energetic recoil of radium D from radium C. Moreover, the number of particles of radium B and radium C expelled from the plate should have been in the ratio of the number of particles of radium B to that of radium C on the plate at the time of an experiment.

The point was tested in the following way:—A disk was suspended 7 millimetres above a strongly active plate with no radium A left on it and exposed to the radiation from it for 10 minutes, the pressure of the air between the plate and disk being maintained at 3·3 mm. of mercury. After exposure, the disk was removed and the time period of the activity received by it tested. As may be seen from fig. 3, the activity was found to fall to half value in about 40 minutes, and this may easily be shown to mean that the numbers of radium B and radium C particles on the disk, when tested, were nearly equal. This was to be expected since, under the experimental conditions, the amounts of radium B and radium C present on the plate must also have been almost equal. It will be noticed that experiments were made with the plate charged with positive and the disk with negative electricity and *vice versa*, and also with the plate and disk at the same potential. This was done because it was thought that possibly the radium C produced from radium B by recoil might become charged after being stopped by the air. If no field existed between the plate and disk, this radium C would diffuse to the disk and plate and to the walls of the containing vessel, whereas in an electric field it might be attracted to one or other electrode. Any such action would have complicated the results, and the effect of an electric field of 340 volts per centimetre was therefore tried. It will be seen, however, that the field had very little effect, the time period of the activity received by the disk being almost independent of the direction of the field, as may be seen from curves II. and III. (fig. 3).

Fig. 3.



The result just arrived at, that radium B and radium C are mechanically projected from the plate as a secondary effect due to the recoil of radium D from radium C, is not a little surprising when we consider the possibility of the formation of a double layer of atoms of active deposit. For the area of the plate was about one square centimetre, and if we take the diameter of a molecule to be 2×10^{-8} cm., we see that it would require about 3×10^{15} active deposit particles to completely cover the plate with a single layer of atoms. But since the quantity of radium C in equilibrium with 1 gram of radium emits $3{\cdot}4\times10^{10}$ α particles per second*, it follows that in this quantity of radium C there are $5{\cdot}77\times10^{13}$ particles. Hence, since in the above experiments there were about the same number of radium B and radium C atoms on the plate, it would have required the amount of these two products in equilibrium with 26 grams of radium to form a single complete layer of atoms on the plate. This is certainly 3000 times as much as was present on the plate in the experiments, and to explain the observed effect it seems necessary to suppose that the active deposit is not evenly distributed over the plate but deposited in heaps at certain places on the plate, the greater portion of which is entirely free from deposit. Whether this localization of the activity is due to the non-uniformity of the electric field during the exposure of the plate to the emanation caused by the unevenness of its surface, or to some other unknown cause, remains a matter for speculation, but the existence of this aggregation of the deposit is certainly striking. It may be mentioned that an active glass surface behaved quite similarly to the platinum surfaces generally used, so that the phenomena do not appear to depend on the smoothness of the surface.

To emphasize the extreme difficulty of getting absolutely pure radium C by the recoil method, an experiment may be cited in which a plate was exposed to a small quantity of radium emanation for some hours, when it was removed and transferred to an evacuated vessel placed in a furnace at 360° C. for nearly half-an-hour to remove emanation. The plate was then tested by its γ radiation and found to be equivalent in activity to ·066 milligram of radium bromide.

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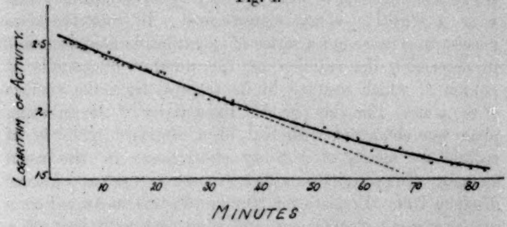
* Rutherford & Geiger, Proc. Roy. Soc. A. vol. lxxxi. p. 141.

A disk was then suspended above the plate *in vacuo*, and after an exposure of 20 minutes, removed and tested by a sensitive α-ray electroscope. The decay of the activity on the disk is shown in fig. 4. It will be seen that the activity

Fig. 4.



at first began to decay with the period of RaC, viz. 19 minutes, which gradually became longer as time proceeded, indicating that even in these circumstances there was present on the disk a small quantity of radium B.

The Proportion of Radium C projected from a Plate by Recoil.

In a previous paper * it was shown that when a disk was exposed above some emanation condensed by liquid air at the bottom of a tube, the amount of radium A and radium B projected on to the disk was about one-eleventh of the quantity which would have reached it if every α particle had been effective in causing a recoil as it left a disintegrating atom. Having regard to the easy absorption of this radiant active matter, this was perhaps as large a fraction as was to be expected, for any slight film of condensed water-vapour or other gas over the surface of the condensed emanation would tend to prevent the escape of the active matter from the surface. A similar result has since been obtained for the recoil of radium B from radium A deposited on a plate. In the case, however, of the recoil of radium C from radium B which is much less energetic, it was at once seen that the fraction of the total number of particles of radium C escaping from an active plate to the total number formed was very small. This was clear without accurate measurement for, whereas the activity of the plate could be measured in a γ-ray electroscope, the activity of a disk exposed to it for 20 minutes could be measured only by careful observations with a sensitive α-ray electroscope. It was therefore necessary to carry out a series of quantitative measurements to determine the fraction of the number of recoils of radium C which resulted in its removal from the surface of the plate. For this purpose, the activity of the radiating plate was compared with that of a standard quantity of radium by means of a γ-ray electroscope in the usual manner. The activity on a disk exposed *in vacuo* at a known distance from the plate for 20 minutes, was measured on a sensitive α-ray electroscope and compared with that of a polonium standard * giving out a known number of α particles per second. In this way the number of α particles emitted from the active disk would be found and the number of atoms of radium C on it deduced.

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* *Loc. cit.*

In a long series of experiments in which the activity of the radiating plate was equal to that of ·2 milligram of radium bromide, the quantity of radium C collected on a disk suspended 2·5 cm. above it for 20 minutes *in vacuo* varied between the equivalent of 5×10^{-7} and $1{\cdot}4 \times 10^{-5}$ milligram of radium bromide. Now, if half the radium C particles produced had been projected upwards from the plate, the disk, which was of diameter 1·7 cm., should have had on it the number of radium C particles equivalent to about 4×10^{-3} milligram of radium bromide at the end of the exposure. Thus the quantity of radium C reaching the disk varied from $\frac{1}{8000}$ to $\frac{1}{300}$ of the total quantity which could have reached it. The smallness of the amount of radium C escaping from an active surface by recoil has recently been noted by Hahn and Meitner †, who obtained little more than one millionth of the total obtainable quantity of radium C on a plate exposed to a surface covered with radium B. This is a far smaller quantity even than we have found; but the experiments of Hahn and Meitner were carried out at atmospheric pressure, which may explain the smallness of the effect obtained by them. The lack of constancy of the amount of radium C emitted from the surface upon which radium B had been deposited is remarkable, seeing that the plates used were always treated in an exactly similar manner. But it was perhaps even more surprising to find that, working with a single surface, the power of emitting radium C varied considerably if the surface were allowed to stand untouched. After making due allowance for the decay of radium B, it was found that an active surface after standing for half an hour or an hour, altered its power of emitting radium C, sometimes becoming a more powerful and on other occasions a less powerful radiator. The cause of these changes in the surface is very obscure, and it is difficult in the present state of our knowledge to advance any satisfactory explanation of it.

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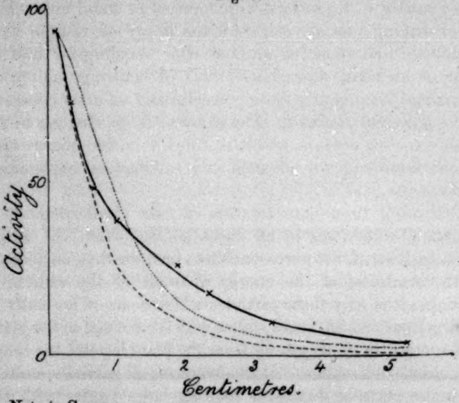
* This standard was kindly lent to us by Dr. Geiger, who determined the number of α particles emitted by it per second.

† Hahn & Meitner, *Phys. Zeitschr.* x. 1909, pp. 697–703.

Returning to a consideration of the small amount of radium C which was in all cases emitted from the active surfaces, it seems not unreasonable to imagine that, on account of the smallness of the energy of recoil of the radium C particles, it is only those particles which come off normally or in directions making small angles with the normal to the plate, that succeed in getting away from the plate beyond the range of molecular attraction. If this were so, all particles emitted from the plate in directions making large angles with the normal to the surface of the plate would be drawn back, and therefore not be detected as radiant matter.

To test the correctness of this view, experiments were undertaken to study the variation with distance of the amount of radium C received by a disk exposed *in vacuo* at different distances from the radiating plate. The apparatus used was that shown in fig. 1. The results obtained are shown in fig. 5 (p. 158). On the same diagram are also plotted the curve showing the variation of activity with distance assuming that particles emitted in all directions from the plate are equally likely to escape from the surface, and also a curve assuming a cosine law for the radiation such as could be applicable to the case of light [*]. It will be seen that the experimental curve agrees with neither of these hypotheses, indicating that the radiation from the plate falls off as the angle which it makes with the normal to the plate increases. The deviation of the experimental curve from that obtained on the supposition that the the particles are radiated in all directions at random seems, however, insufficient to entirely account for the smallness of the fraction of radium C particles which succeed in getting free from the radiating surface.

Fig. 5.



Note to Curve :—

Line thus ——— represents the Experimental Curve.

" " ------ " " Inverse Square Law Curve.

" " ·········· " " Cosine Law Curve.

The Absorption of Radium C by Air.

When radium B is transformed into radium C, the process is supposed to be accompanied only by the emission of β particles of low velocity [†]. It therefore follows that the energy possessed by a radium C atom after recoil should be exceedingly small compared with that of an atom of radium B when produced from radium A or with that of any atom which recoils as the result of the emission of an α particle. On this account, the power of penetrating matter possessed by an atom of radium C should be very much smaller than that, say, of an atom of radium A or radium B when projected from a disintegrating atom. For if we take the mass of an atom of radium A to be 218 and that of an α particle to be 4, and its velocity to be $1 \cdot 77 \times 10^9$ centimetres per second *, then, from the equation of momentum, the velocity with which the particle of radium B must be travelling after recoil will be $3 \cdot 25 \times 10^7$ centimetres per second. It is evident that an atom of radium A when formed from the emanation will recoil with approximately the same velocity. On the other hand, since the mass of an atom of radium B is 214 and that of a β particle $\frac{1}{1700}$, then, if we assume the velocity of the β particle from radium B to be 10^{10} centimetres per second †, the velocity of expulsion of the radium C atom formed by the process will be $2 \cdot 75 \times 10^4$ centimetres per second. It therefore follows that an atom of radium B will recoil with $1 \cdot 39 \times 10^6$ times the energy of an atom of radium C. The penetration of matter by radium C should therefore be less than one-millionth of that of radium B, if the power of an atom of penetrating matter is proportional to its energy.

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[*] These curves have been plotted from values obtained by Mr. H. Bateman (Phil. Mag. Dec. 1909).

[†] H. W. Schmidt, *Phys. Zeitschr.* vi. p. 897, and *Annalen der Physik*, xxi. p. 609.

Now the latter radiation is exceedingly easily absorbed, being cut down to one-tenth of its initial value by passing through 6·5 centimetres of air at a pressure of one millimetre of mercury. It might therefore be expected that even a minute quantity of air would be sufficient to entirely stop radium C from escaping any distance from a deposit of radium B. This is, however, not the case; for the radium C expelled by recoil from radium B can penetrate appreciable quantities of air before being stopped, and it was even found possible to measure its absorption by air, though the experiments were difficult and the accuracy of the results not great.

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* Rutherford, Phil. Mag. Oct. 1906.

† There appear to be no available data regarding the velocity of the β particles emitted by radium B; but the value assumed is such as to give an upper limit to the velocity of expulsion of radium C.

The apparatus used was that shown in fig. 1, p. 149. In view of the very large absorption to be expected, and on account of the necessity of working with small quantities of radium B in order to get pure radium C projected on to the disk B, it was at first thought desirable to bring the disk close to the radiating plate. The first experiments were therefore made with the disk suspended 2 millimetres above the active plate, and the radiation received by the disk after an exposure of 20 minutes to the plate measured, the pressure of the air in the vessel V being varied. It was, however, soon found that working in this way very irregular results were obtained, and it seemed that the disturbances were produced by secondary causes due to the diffusion of the radium C particles when stopped by the air; for if a certain proportion of the atoms of radium C projected from the active plate are stopped by impact with air molecules before reaching the disk suspended above, then it becomes a matter of chance whether such particles diffuse on to the disk or back again on to the plate. In fact a small current of air passing between the plate and disk might make all the difference in deciding whether the disk received much or little radium C from the plate. The distance between the disk and plate was therefore increased as far as was possible without rendering the amount of radium C reaching the disk unmeasurably small. With the disk 2·5 centimetres from the plate, and using the active deposit equivalent to ·2 milligram of radium bromide on the surface of the plate, it was found just possible to make the required measurements. Even in these circumstances the experiments were difficult to perform with any precision, for, as has been mentioned above, the amount of radium C expelled from a plate coated with a definite quantity of radium B is a variable quantity, and it was therefore necessary not only to test the quantity of active deposit on the plate before starting an experiment but also to determine its power of radiating radium C. Unfortunately, the measurements were still further complicated by the changes in the radiating power of the surface when left to stand: the surface had therefore to be tested after as well as before an experiment, to make sure that it had not changed enough to completely vitiate the results. Surfaces have never yet been obtained which maintained their power of radiating radium C perfectly satisfactorily, but the changes were erratic, being sometimes very large and sometimes comparatively small. Only those experiments were made use of in which the changes of surface were reasonably small: other experiments were rejected.

The method of conducting an experiment consisted in exposing a platinum plate, mounted on an iron disk to give it rigidity, to some radium emanation for several hours. The plate was then removed from the emanation and allowed to stand for three-quarters of an hour to allow radium A to decay to an inappreciable amount, the emanation adhering to the plate being driven off in the manner already described. The activity of the plate having been measured by a γ-ray electroscope, it was mounted on the support A and the disk B was suspended 2·5 cm. above it to receive the radium C expelled from the plate. The air in the vessel V was then pumped out as quickly as possible by a Gaede pump, and after 20 minutes the disk B was removed and tested by a sensitive α-ray electroscope. It was noticed that both sides of the disk were radioactive when tested, though the activity of the front side of the disk which had faced the radiating plate was always greater than that of the back. Now no active matter could have reached the back by direct radiation, and its activity was at first attributed to traces of emanation left on the plate and gradually escaping from it. But it was subsequently shown that this was not the case; for when the disk was exposed to an active plate for a long time, so that all the radium B and radium C present had decayed, neither side of the disk showed the slightest activity. No doubt the real explanation of the effect is that a portion of the radium C emitted by the plate is stopped before reaching the disk by the residual air left even at very low pressures and then diffuses round to the back of the disk. In order to find the amount of radium C directly shot on to the disk, the activity of the back of the disk was subtracted from that on the front.

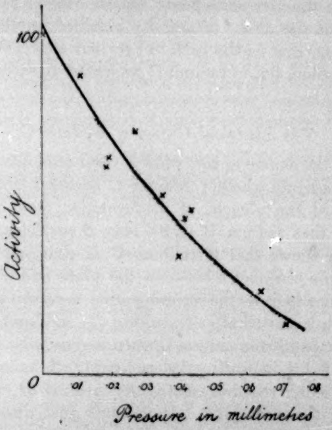
After exposing a disk in this way another similar one was then suspended above the plate, and the pressure of the air in V adjusted to any desired value which was read on a McLeod gauge. After 20 minutes this disk was removed and tested, and a third one was inserted and exposed to the radiation for 20 minutes *in vacuo*. Allowing for the decay of radium B on the plate, it was easy to calculate the amount of radium C which should have reached this third disk in terms of the quantity which reached the first disk exposed. If the quantity of radium C found on the disk proved to be very different from the calculated amount, the experiment was rejected ; in other cases, the mean of the two readings obtained *in vacuo* was taken. In this way it was possible to get some idea of the law of absorption of radium C by air. The results are given in Table I. and fig. 6, below. The absorption at different pressures can be seen from the last column, which gives the quantity of radium C reaching the disk, the mean of the two vacuum readings being taken as 100.

TABLE I.

Pressure, in millimetres of Mercury.	Activity in Arbitrary Units.			Percentage of the Mean of the two Vacuum Readings.
	Front.	Back.	Front–Back.	
·0114	85	36	49	88
·019	1040	182	858	61
·020	252	56	196	64
·028	226	61	165	72
·036	347	65	282	53
·041	186	66	120	35
·043	88	44	44	46
·045	333	145	188	49
·066	89	31	57	24
·073	90	38	51	14

Although the results are not very consistent, the order of magnitude of the absorption can be seen from the numbers given. By a comparison with the absorption of the radiant matter from the emanation condensed at the bottom of a tube in liquid air, studied in a previous paper, it will be seen that radium A and radium B, when expelled as the result of a recoil from an α particle, can penetrate only about 40 times as much air without being stopped as radium C when it recoils as the result of the emission of a β particle from radium B. Now it has already been calculated that the energy possessed by radium C should be less than one-millionth of that possessed by radium A or radium B after recoil. The order of magnitude of the absorption of radium C is therefore far smaller than was to be anticipated. It is difficult to explain this discrepancy; if an α particle of slow velocity were expelled during the transformation of radium B into radium C, the difficulty would be removed. But it must be admitted that the possibility of the existence of an undetected α particle in this transformation seems remote in view of other evidence.

Fig. 6.—Absorption of RaC by Air.



There still remains the question whether all the excess of activity on the front of the disk over that on the back at the higher pressures is due to radium C projected directly on to it, or whether some of it finds its way there by diffusion after having been stopped by the residual gas left even at the highest obtainable vacuum. To test this possibility the following experiment was made:—According to the procedure already detailed, a brass disk was fixed 1·4 centimetre above a radiating surface, but shielded from direct radiation by a copper screen placed between the surface and the disk so that any activity obtained on the disk must have reached it by diffusion. The exposure lasted half-an-hour *in vacuo*, and on removal the disk was found to have a very small activity both on the front and back, the activity on the two sides being almost exactly equal. This showed that the excess of activity obtained on the front of the disk over that on the back in previous experiments is not due to diffusion, but to radium C projected directly on to the disk.

The Electrical Charge of Radium C.

During the course of this work several attempts have been made to ascertain whether radium C becomes charged with electricity at the moment of its formation from radium B. Assuming that radium B emits only β particles, a simple calculation shows that if radium C is charged, a field of 500 volts per centimetre between the plate and disk (fig. 1) should suffice to cause the recoiling atom to return to the plate from which it started after travelling $\frac{1}{400}$ millimetre.

All attemps to stop radium C when it recoils by an electric field have failed, indicating that radium C when formed from radium B either remains electrically neutral or, as is suggested by the absorption experiments just described, the energy of a recoiling atom of radium C is, for some unknown reason, greater than has been supposed.

We have again to thank Professor Rutherford for his kindness in supplying us with the necessary radium emanation and also for his interest in this work.

ABSTRACT.

It has been shown in a previous paper, that during a radioactive transformation involving the expulsion of an a particle, the residue of the atom from which the a particle has been expelled recoils in an opposite direction to that in which the a particle is emitted, and can travel a considerable distance through a gas if the pressure is sufficiently low. A similar effect was also demonstrable in the case of the transformation

of radium B into radium C, although this transformation is supposed to be accompanied by only β rays. The phenomena associated with this recoil are studied in this paper.

In the first place it was found that it was only in certain circumstances that pure radium C free from radium B was projected from a plate coated with radium B and radium C. The reasons for this were two-fold: firstly, only a very small fraction of the radium C produced from radium B escaped from the plate so that even a small quantity of radium A on the plate was capable of emitting a comparatively large quantity of radium B. Secondly, the active deposit on a plate appears to be concentrated into heaps, so that radium C in breaking up mechanically carried with it some radium B. If, however, sufficient time is allowed after removing a plate from the emanation for radium A to completely decay and if, further, sufficiently small quantities of deposit are used to avoid the formation of heaps, practically pure radium C is emitted.

The law according to which the radiation fell off with distance was also studied, and it was found that radium C is not emitted from an active plate equally in all directions, a greater quantity being emitted normally to the plate than in directions making an angle with the normal.

The absorption by air of radium C when it recoils from radium B was investigated. Although the experiments were difficult to perform owing to the ease with which the radium C is stopped by the air and owing to uncertain changes in the radiating power of the plate, it was possible to get some estimate of the penetration of radium C. It was found that about half the radium C projected from a plate was stopped by 2·5 cms. of air at a pressure of 0·04 mm. mercury. Since radium B emits only β particles, the energy of recoil in this case should be less than one-millionth of the energy of recoil in a transformation in which an α particle is emitted. The fact that the penetration of radium C when it recoils is as much as one-fortieth of that previously found for radium A and radium B is therefore surprising.

All attempts to stop radium C by an electric field when it recoils have failed, indicating that radium C when formed from radium B either remains electrically neutral or, as is suggested by the absorption experiments described, the energy of a recoiling atom of radium C is for some unknown reason greater than has been supposed.

XII *The Sun's Motion with respect to the Æther.*

By C. V. BURTON, *D.Sc.*[*]

1. SOME time ago, in a letter to 'Nature,' I ventured to point out that the Sun's velocity with respect to the æther could be deduced from the observed times of the eclipses of Jupiter's satellites. At that time it was unknown to me that the following remark on the same subject had been made by Maxwell[†]:—

> "The only practicable method of determining directly the relative velocity of the æther with respect to the solar system is to compare the values of the velocity of light deduced from the observation of the eclipses of Jupiter's satellites when Jupiter is seen from the earth at nearly opposite points of the ecliptic."

If for brevity we speak of the motion of the æther with respect to the solar system as a *wind*, the essence of the matter might be expressed by saying that light from Jupiter would reach us more quickly when that planet was to windward of us than when it was to leeward; the wind-velocity being in the former case added to, and in the latter case subtracted from, the velocity of propagation. Maxwell's reference to Jupiter as "seen from the earth at nearly opposite points of the ecliptic" is evidently a mere slip. The context makes it clear that the very simple essentials of the problem had been fully realized.

2. Though some thirty years have elapsed since Maxwell wrote, attention does not seem to have been directed to the subject in any effective degree; yet from the standpoint of electromagnetic theory, great interest attaches to any physically possible method of determining motion with respect to the æther; while from an astronomical point of view the results to be looked for are no less important. Unfortunately, owing to the small inclination of Jupiter's orbit to the ecliptic, we are virtually restricted to a problem in two dimensions. The present would seem to be a favourable

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[*] Read November 26, 1909.

[†] *Encyclopædia Britannica*, 9th ed., article "Ether."

time for directing the attention of astronomers and physicists to these questions anew; on the one hand because Prof. Kapteyn's discovery of two principal star-drifts makes it important to determine the motion of each drift with respect to the æther, so far as we can; and on the other hand because the material needed for the computations will soon be available. The Harvard photometric observations of the eclipses have been very fully discussed by Prof. R. A. Sampson, in a memoir which is about to appear as vol. lii. pt. ii. of the Harvard Annals*.

3. Let the origin of coordinates coincide always with the sun's centre, the coordinate axes being in directions fixed in space. At any time t let the coordinates of Jupiter's satellite I. be (x', y', z') and those of the earth (x, y, z); let the velocity of the sun with respect to the æther have components (a, b, c) in the directions of the coordinate axes, the assumption being made that this velocity does not vary sensibly within the period covered by the observations considered.

4. Let an eclipse of satellite I. be *observed* at time t'; then the eclipse actually occurred at some previous time $t=t'-\tau$, when the coordinates of the satellite were

$$\{x'-(a+\dot{x}')\tau,\quad y'-(b+\dot{y}')\tau,\quad z'-(c+\dot{z}')\tau\}\,; \quad . \quad (1)$$

the axes of reference being here understood to have the same position *in the æther* as our heliocentric axes have at time t. In the interval of time τ, therefore, light has travelled through the æther from (1) to a point whose coordinates, referred to the same axes, are (x, y, z). It may be provisionally assumed that $a+\dot{x}'$, $b+\dot{y}'$, $c+\dot{z}'$ are small compared with V, the velocity of light; the distance between (1) and (x, y, z) is then approximately

$$r+\frac{\tau}{r}\{\xi(a+\dot{x}')+\eta(b+\dot{y}')+\zeta(c+\dot{z}')\}\,; \quad . \quad . \quad (2)$$

where

$$\xi,\ \eta,\ \zeta \equiv x-x',\ y-y',\ z-z', \text{ and } r^2 \equiv (x-x')^2+(y-y')^2+(z-z')^2. \quad (3)$$

5. Since (2) has to be equated to Vτ, each observation

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* Since this note was written Prof. Sampson has favoured me with some advance sheets of his work, wherein are tabulated just those residual errors from which the Sun's velocity has to be computed.

furnishes an equation of the form

$$\mathrm{V}\tau-r-\tau/r\,.\,\{\xi(a+\dot{x}')+\eta(b+\dot{y}')+\zeta(c+\dot{z}')\}=0\,;$$

or, remembering that r/τ is very approximately equal to V,

$$\xi(a+\dot{x}')+\eta(b+\dot{y}')+\zeta(c+\dot{z}')-\mathrm{V}(\mathrm{V}\tau-r)=0. \quad . \quad (4)$$

Now $(\dot{x}', \dot{y}', \dot{z}')$ is the resultant of Jupiter's orbital velocity and of the velocity of satellite I. in its orbit around the planet. At the time of an eclipse it may amount, perhaps, to 22 kilom. per second, which is about half the smallest possible estimate of the probable error in the most favourably conditioned velocity-component (a) [see (11) below]. Moreover, the error introduced into a, b, or c by neglecting $\dot{x}'$, $\dot{y}'$, and $\dot{z}'$ would be a very small fraction of 22 kilom. per second, and accordingly equations (4) may be simplified by the omission of $\dot{x}'$, $\dot{y}'$, $\dot{z}'$.

6. These simplified equations can be treated by the method of least squares. In this preliminary discussion, all the observations are taken to be of equal weight, so that we have simply to make the sum of the squares of the left-hand members of these equations a minimum with respect to a, b, and c. Thus

$$\left.\begin{array}{l} a\Sigma\xi^2+b\Sigma\xi\eta+c\Sigma\zeta\xi=\mathrm{V}\Sigma\xi(\mathrm{V}\tau-r) \\ a\Sigma\xi\eta+b\Sigma\eta^2+c\Sigma\eta\zeta=\mathrm{V}\Sigma\eta(\mathrm{V}\tau-r) \\ a\Sigma\zeta\xi+b\Sigma\eta\zeta+c\Sigma\zeta^2=\mathrm{V}\Sigma\zeta(\mathrm{V}\tau-r) \end{array}\right\}; \quad . \quad . \quad (5)$$

and from these equations values for a, b, c can be at once written down.

7. Before discussing the choice of axes, it will be convenient to obtain general expressions for the probable errors of a, b, and c. The residuals $\mathrm{V}\tau-r$ may be regarded as the sole source of error, since all the other quantities involved in (5) are known with ample accuracy for our purpose. In dealing with the various errors which may affect the values adopted for the quantities $\mathrm{V}\tau-r$, let the suffix $_0$ distinguish true values, while the suffix $_s$ indicates those quantities which are variable from one $\mathrm{V}\tau-r$ to another. Thus let

$$\mathrm{V}=\mathrm{V}_0+v, \quad r=r_0+kr_0; \quad . \quad . \quad . \quad . \quad (6)$$

so that v is the error in the accepted value V for the velocity of light, and k is the proportional error in the value assigned to the solar parallax. (The angular magnitudes involved in our estimate of r can be so accurately ascertained that no appreciable error can arise from them.)

8. Again, let $(\tau_0)_s$ denote the time actually taken by light to travel from the satellite at mid-eclipse to the observer. Then, $(\tau_0)_s$ being the difference between the apparent and the actual time of mid-eclipse, the value τ_s which is available to us will be liable to differ from $(\tau_0)_s$ owing to three sources of error:—

(i.) The irregularity of Jupiter's surface causes the actual eclipses to succeed one another at *intervals* of time which are sensibly discrepant from those calculated theoretically. Thus τ_s is affected by an error of prediction which is periodic or quasi-periodic* in character, and which we denote by χ_s.

(ii.) There will also necessarily be some slight uncertainty as to the epoch of the actual eclipses, and τ_s will on this account be liable to a small constant error (γ).

(iii.) In determining the apparent time of mid-eclipse, an error of observation is incurred; let the error thus introduced into the value of τ_s be called v_s; we have accordingly

$$\tau_s = (\tau_0)_s + \chi_s + \gamma + v_s. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

9. For any single eclipse-observation, to a sufficient approximation,

$$V\tau - r = V_0\tau_0 - r_0 + V(v_s + \chi_s + \gamma) + \tau v - kr. \quad . \quad . \quad (8)$$

Without, in the first place, taking account of the motion of the solar system through the æther, we may make γ very small by making use of a number of eclipse-observations extending over at least one Jovian year. Moreover, γ only enters into our results in association with factors which tend approximately to disappear in the long run; its effect may therefore be disregarded. For similar reasons our final results will not be appreciably affected by the small errors v and k. (When the numerical values of a, b, c have been found from (5), it will be possible, without much additional labour, to determine more definitely the influence of given errors γ, v, k on the values found for a, b, c, as well as the probable errors in a, b, c arising from an assumed "all round" value for the probable observational error corresponding to a single eclipse. For the present we must be content with a more tentative estimate of the accuracy to be expected.)

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* Though Prof. Sampson tells me that no period has been recognized in the residual errors.

10. Finally, since the residual effects for which we are searching depend on the direction of the line drawn from the earth to Jupiter, and have thus a principal period of about 12 years, it is especially important for our purpose that the Jovian system should be free from any influence having an approximately equal period, and capable of affecting the times of eclipse of the first satellite. Prof. Sampson kindly went into this question with me, and found that no disturbing influence of the period in question was to be feared. We may accordingly confine our attention (in this preliminary discussion at least) to the errors typified by $\chi_s + \upsilon_s$.

11. Referring now to equations (5), we find that the error $\chi_s + \upsilon_s$ in a single observation gives rise to an error in a equal to

$$\frac{V^2(\chi_s+\upsilon_s)}{D}\begin{vmatrix} \xi_s & \Sigma\xi\eta & \Sigma\zeta\xi \\ \eta_s & \Sigma\eta^2 & \Sigma\eta\zeta \\ \zeta_s & \Sigma\eta\zeta & \Sigma\zeta^2 \end{vmatrix}\text{; where } D \equiv \begin{vmatrix} \Sigma\xi^2 & \Sigma\xi\eta & \Sigma\zeta\xi \\ \Sigma\xi\eta & \Sigma\eta^2 & \Sigma\eta\zeta \\ \Sigma\zeta\xi & \Sigma\eta\zeta & \Sigma\zeta^2 \end{vmatrix}$$

Thus if ϵ represents the all-round probable value of the error $\chi_s + \upsilon_s$ for a single eclipse, we shall have

$$\text{p.e. in } a = \frac{V^2\epsilon}{D}\left\{\Sigma\begin{vmatrix} \xi_s & \Sigma\xi\eta & \Sigma\zeta\xi \\ \eta_s & \Sigma\eta^2 & \Sigma\eta\zeta \\ \zeta_s & \Sigma\eta\zeta & \Sigma\zeta^2 \end{vmatrix}^2\right\}^{\frac{1}{2}}$$

$$= V^2\epsilon D^{-\frac{1}{2}}(\Sigma\eta^2\Sigma\zeta^2 - \Sigma\eta\zeta\,.\,\Sigma\eta\zeta)^{\frac{1}{2}}; \quad . \quad . \quad . \quad (9)$$

as is found on expanding and simplifying. The probable errors in b and c can of course be similarly expressed.

12. In order to obtain a preliminary notion of the conditions of the problem without undue labour, I have considered a simplified system which is not strictly speaking a dynamically possible one, but which represents the actual system

sufficiently nearly for the purpose in view. The observer is supposed to be carried uniformly round the sun in a circular orbit whose radius R is equal to the mean radius of the earth's orbit, the time of revolution being one year. The Jovian system is supposed to move in a very slightly elliptic orbit, with the sun at the *centre* of the ellipse, the inclination, ϕ, of this orbit to that of the observer being 1° 18′ 41″, which is the actual inclination of Jupiter's orbit to the plane of the ecliptic. The excentricity of the modified orbit of Jupiter is such that its projection on the plane of the observer's orbit is a circle, this circle being uniformly described in 11 years 315 days, and having a radius R′ equal to the mean radius of Jupiter's actual orbit. To take account of the fact that eclipse observations are impracticable for some time before and after Jupiter is in conjunction, an eclipse is considered as possibly observable only when the angle earth-sun-Jupiter lies between the limits $\pm\psi$ ($\psi=102°$, say). Between these limits, eclipse observations are taken to be uniformly frequent and uniformly weighty. With the axis of x through the node of Jupiter's (modified) orbit, the axis of y in the perpendicular direction in the plane of the ecliptic, and the axis of z perpendicular to the ecliptic, the results found are

$$\left.\begin{aligned}(\text{p.e. in } a)^2 &= \frac{2V^4\epsilon^2}{n}\cdot\frac{1}{R^2-2R'R\sin\psi/\psi+R'^2}\\(\text{p.e. in } b)^2 &= \frac{2V^4\epsilon^2}{n}\cdot\frac{1}{R^2(1-\sin^2\psi/\psi^2)}\\(\text{p.e. in } c)^2 &= \frac{2V^4\epsilon^2}{n}\cdot\frac{R^2-2R'R\sin\psi/\psi+R'^2}{\sin^2\phi R^2R'^2(1-\sin^2\psi/\psi^2)}\end{aligned}\right\} . \qquad (10)$$

13. In these expressions $R=148\times10^6$ kilom., $R'/R=5{\cdot}2$, $V=3\times10^5$ kilom. per second, $\phi=1°\ 18'\ 41''$; n, the number of available observations, is about 330; and ϵ, the "probable" discrepancy between the observed and calculated time of an eclipse, is estimated by Prof. Sampson at about 4·5 seconds. The numerical values thus derived from (10) are

$$\left.\begin{aligned}&\text{p.e. in } a=44{\cdot}0 \text{ kilom. per second}\\&\quad,,\quad,,\ b=237\quad ,,\qquad ,,\\&\quad,,\quad,,\ c=10{,}000\quad ,,\qquad ,,\end{aligned}\right\} . \ . \ . \qquad (11)$$

14. The determination of the velocity-component *c* (perpendicular to the plane of the ecliptic) is so badly conditioned that the investigation can hardly be considered to afford any light on that point. It may be admitted as probable that the velocity of the solar system through the æther is very far below 10,000 kilom. per second; otherwise we should have to suppose (for example) that practically all stars whose radial velocities have been measured are moving through the æther with velocities of thousands of kilometres per second. This certainly appears unlikely, though perhaps the possibility ought not to be too lightly denied. An indirect argument against a very high velocity of the solar system may be derived from other considerations. For we know that the velocity components in the plane of the ecliptic must be relatively moderate, otherwise there would be a marked anomaly, having a period of about 12 years, in the observed times of the eclipses of Jupiter's satellites*, and *à priori* it is unlikely that the velocity of the solar system should be nearly in some arbitrarily assigned direction, such as that perpendicular to the plane of the ecliptic.

15. On the other hand, if we begin by admitting complete ignorance regarding the velocity-components to be determined—save only our assumption that they are small compared with the velocity of light—then 10,000 kilom. per second will represent approximately the probable error in the value of the component *c*; and *in general* the determination of the component of velocity resolved in any direction in space will be affected by a large probable error arising from the uncertainty in the value of *c*. On this view, the exact choice of coordinate axes becomes of importance, the relatively large probable error in *b* given by (11) being due to the influence of the error in *c*. So long as the axes remain as specified in § 12, this unduly large error in *b* can only be avoided (if at all) by an excessive arbitrary weighting of the observations, so designed as to secure the vanishing of the coefficient of *c* in each of the first two of equations (5); these two equations being then solved for *a* and *b*.

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* It may be hoped that the investigation now proposed will indicate these components more definitely, or at least a superior limit to them.

16. In the simplified problem of § 12, it appears that, without special weighting of the observations, the probable error in the velocity-component b can be greatly reduced by a slight change of axes. The axis of x remaining unchanged, the axes of y and z are to be rotated in their own plane through a small angle; and if the velocity-components along the new axes are called a, b_1, c_1, it is found that the probable error in b_1 is a minimum when the axis of y lies very nearly in the plane of Jupiter's (modified) orbit. If we take that plane as the plane of xy, the probable errors, corresponding to our simplified problem, are:

$$\left.\begin{array}{l}\text{p.e. in } a = 44{\cdot}0 \text{ kilom. per second} \\ \phantom{\text{p.e.}} \text{,, ,, } b_1 = 45{\cdot}5 \quad \text{,,} \qquad \text{,,} \\ \phantom{\text{p.e.}} \text{,, ,, } c_1 = 10{,}000 \quad \text{,,} \qquad \text{,,}\end{array}\right\}; \quad . \quad . \quad (12)$$

using 330 observations, each with a probable error of 4·5 seconds.

17. In the absence of definite reasons to the contrary, it would be natural to take the plane of the ecliptic as one of the coordinate planes, but it appears that, by taking the plane of Jupiter's orbit instead, we can avoid increasing the probable error of one velocity-component (b or b_1) by an unknown amount, and this without adding seriously to the labour of computation. In the actual problem, therefore, it will probably be advisable to determine the values of two velocity-components in the plane of Jupiter's orbit, using for this purpose equations corresponding to the first two of (5), and causing the coefficients $\Sigma\zeta\xi$, $\Sigma\eta\zeta$ to disappear as nearly as may seem necessary by a *moderate* special weighting of the observations. We may then proceed more tentatively, *on the assumption that* c *is moderate*, to identify the velocity-component b_1 so found with the corresponding component b in the plane of the ecliptic. If, for example, on general grounds, and in the absence of any positive information, we assume $c = 0 \pm 500$ kilometres per second, the probable error in b thus arising will be approximately 12 kilom. per second, and the total probable error in b will be about $\sqrt{(45{\cdot}5^2 + 12^2)} = 47$ kilom. per second.

I hope very shortly to consider this problem more in detail; meanwhile I wish to thank Prof. Sampson for his kind and most helpful advice.

Boar's Hill, near Oxford,
9 October. 1909.

ABSTRACT.

Nothwithstanding the well-known "principle of relativity," it is theoretically possible to determine the motion of the solar system with respect to the æther from observations of the eclipses of Jupiter's satellites; and the possibility was indicated by Maxwell some 30 years ago. For convenience, the motion of the æther with respect to the sun may be called a wind, and the method proposed is based on the consideration that the tidings of an eclipse will travel towards us more rapidly when the Jovian system is to windward of us than when it is to leeward. The residual discrepancies between the observed and calculated times of eclipses have to be analysed for systematic differences depending on the direction in space of the straight line drawn from the earth to Jupiter, and formulæ are given for finding by the method of least squares the most probable values of a, b_1, c_1, the components of the Sun's velocity with respect to the æther. The material available is to be found in Prof. R. A. Sampson's discussion of the Harvard photometric eclipse-observations; about 330 eclipses of Jupiter's satellite I being included.

In order to obtain a preliminary notion of the accuracy to be expected, a simplified system has been considered in which (for one thing) the eccentricity of the orbits was virtually neglected; and it appears that some advantage is to be gained by taking the plane of Jupiter's orbit, rather than the ecliptic, as one of the coordinate planes. The axis of x is drawn from the Sun's centre through the node of Jupiter's orbit, the axis of y lying also in that orbit, and the axis of z being perpendicular thereto. Taking 4·5 seconds as the "probable" discrepancy between theory and observation for a single eclipse, the following preliminary estimates are obtained:—

Probable error in $a = 43{\cdot}6$ km. per second.
" " $b_1 = 45{\cdot}6$ " "
" " $c_1 = 10{,}000$ " "

The determination of the component perpendicular to Jupiter's orbit is perhaps too badly conditioned to be worth considering; the two components in the plane of Jupiter's orbit can be much better computed, and even if the velocities found do not exceed the probable limits of error, an upper limit to the numerical values of those components can be assigned.

Fig. 2.

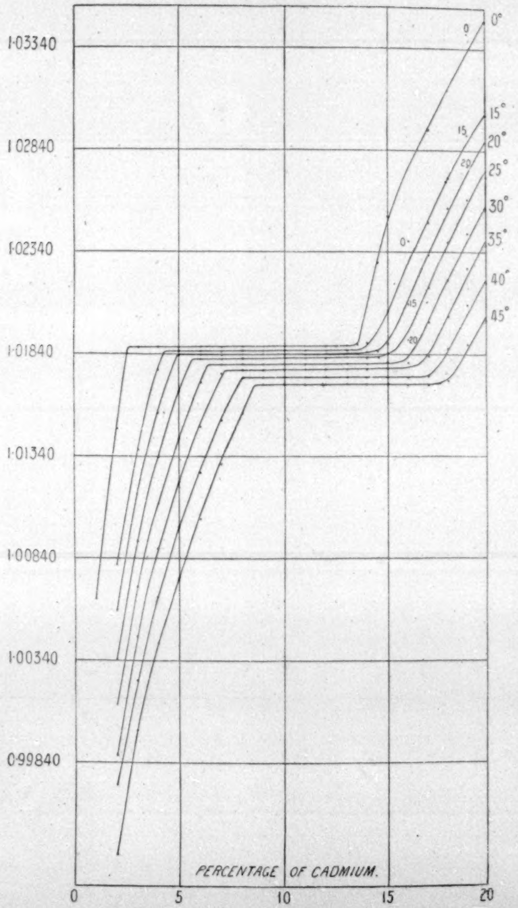


Fig. 3.

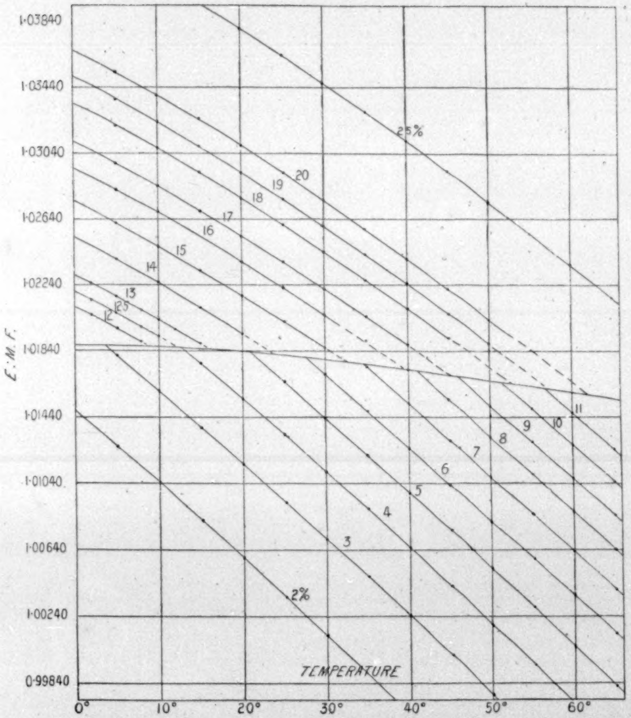
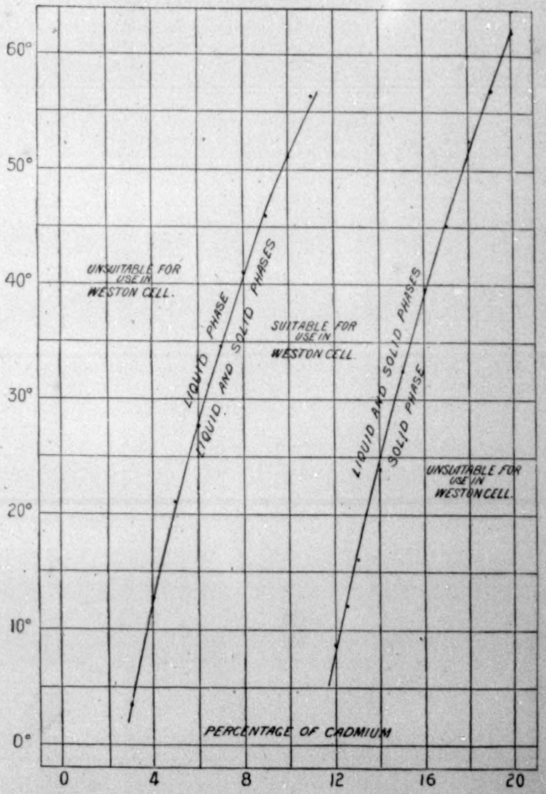
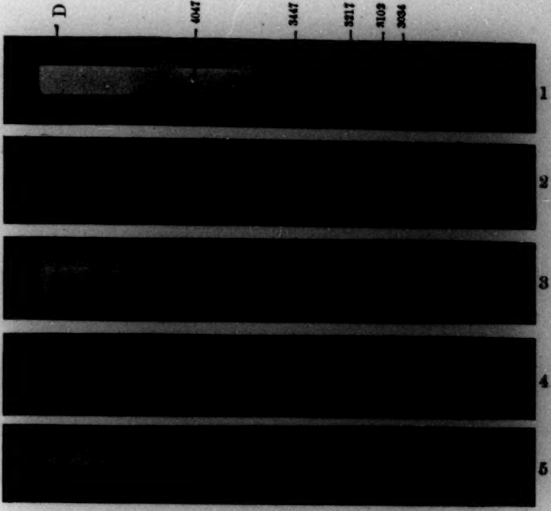


Fig. 4.

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3103
3034
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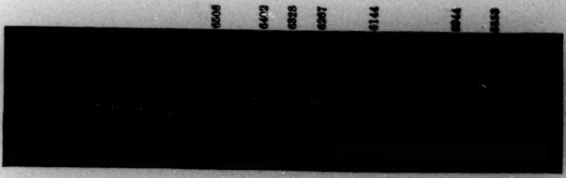
Fig. 1.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6402
6328
6267
6144
5944
5852

Fig. 2.

XIII. *Saturation Specific Heats, &c., with van der Waals' and Clausius' Characteristics.* By Robert E. Baynes, *M.A.*[*]

[Plate IV.]

Over twenty years ago, and shortly after reading Planck's 1881 Memoir "Die Theorie des Sättigungsgesetzes" in vol. xiii. of Wiedemann's *Annalen*, I calculated in the following manner the values for different temperatures of the latent heat of vaporization and of the saturation specific heats on the assumption of van der Waals' equation, as also the values of other magnitudes. I did not publish my results as the equation does not represent actuality; but such of them as have not since been published by others I desire now to record, since they contrast very greatly with the results of similar calculations which I have lately made on the assumption of Clausius' characteristic.

The thermal capacities of a fluid with van der Waals' characteristic

$$(p+av^{-2})(v-b) = \mathrm{R}t$$

being defined by

$$d\mathrm{H} = kdt + ldv = \mathrm{K}dt + \mathrm{L}dp,$$

their values are

$$k = \mathfrak{k}, \qquad l = \frac{\mathrm{R}t}{v-b}, \qquad \mathrm{K} = \mathfrak{k} + \frac{\mathrm{R}}{1-2a(v-b)^2/\mathrm{R}tv^3},$$

$$\mathrm{L} = -\frac{v-b}{1-2a(v-b)^2/\mathrm{R}tv^3},$$

where $\mathfrak{k}$ is in the general case a function of t.

Their further calculation is simplified by using the characteristic in its 'reduced' form

$$(\pi+3\nu^{-2})(3\nu-1) = 8\tau,$$

in which the units employed are the critical values P, V, T of the pressure, volume, and temperature respectively, in terms of which we have

$$a = 3\mathrm{PV}^2, \qquad b = \tfrac{1}{3}\mathrm{V}, \qquad \mathrm{R} = 8\mathrm{PV}/3\mathrm{T};$$

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[*] Read January 21, 1910.

and if, to simplify our calculations, we introduce * a new variable μ defined by

$$\mu = +\sqrt{\{(3\nu-1)^2-4\pi\nu^3\}},$$

which will be real in all the cases we need consider, and for shortness of expression put

$$\begin{array}{lll} A \equiv 3\nu-1, & B \equiv \tfrac{1}{2}(3\nu-1-\mu), & C \equiv \tfrac{1}{2}(3\nu-1+\mu), \\ H \equiv \tfrac{1}{2}(9\nu-1+\mu), & D \equiv \tfrac{1}{2}(3\nu+1-\mu), & E \equiv \tfrac{1}{2}(3\nu+1+\mu), \\ & F \equiv \tfrac{1}{2}(3-3\nu-\mu), & G \equiv \tfrac{1}{2}(3-3\nu+\mu), \end{array}$$

we have

$$\pi = BC/\nu^3, \qquad \tau = ADE/8\nu^3,$$

$$l/P = DE/\nu^3, \quad (K-\mathfrak{k})/R = DE/FG, \quad L/V = -ADE/3FG.$$

To determine the variations of the thermal capacities we may plot their values for given values of τ against the values of π or ν. We easily see that the $l\pi$, $L\pi$, $L\nu$ curves may present singularities while the $l\nu$ curves do not; that the value of $K-\mathfrak{k}$, which is positive except when ν lies between $1+\frac{1}{3}\mu$ and $1-\frac{1}{3}\mu$, *i.e.* except when ν lies between the largest pair of the roots of the equation $4\tau\nu^3 = (3\nu-1)^2$ which are all real when $\tau<1$, is always a maximum when $\nu=1$ whatever the value of τ, the maximum value for τ being $R\tau/(\tau-1)$ and the corresponding reduced pressure being $4\tau-3$; that $K-\mathfrak{k}$ has also a minimum value R when $\nu=\frac{1}{3}$, this being also its value for any value of τ when $\nu=\infty$.

If the isopiestic for π cuts the isothermal for τ in three points and ν, ν'', ν' denote the corresponding volumes in ascending order of magnitude, then, if μ is defined as above strictly with reference to ν,

$$\nu'' = \nu/C, \qquad \nu' = \nu/B,$$

and the heat-capacities corresponding to ν', expressed in terms of ν, are given by

$$l'/P = ABD/\nu^3, \qquad (K'-\mathfrak{k})/R = AD/\mu G,$$
$$L'/V = -ADE/3\mu BG.$$

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* This variable was also employed by Ritter, *Wien. Sitz.-Ber.* July 1902.

Case of Saturation.

Now, if π is the *saturation*-pressure at τ, we have also the relation

$$3(\pi+3/\nu\nu')(\nu'-\nu) = 8\tau \log\{(3\nu'-1)/(3\nu-1)\},$$

so that, in the case of saturation, ν and μ are connected by the relation

$$\log(\mathrm{E}/\mathrm{AB}) = 3\nu\mathrm{GH}/\mathrm{ADE},$$

which may be looked upon as the equation of the liquid side of the connode or boundary curve.

[The equation of the vapour side of the connode is of the same form but with the sign of μ changed; for we similarly obtain

$$\log\frac{3\nu'+1-\mu'}{(3\nu'-1)(3\nu'-1+\mu')} = \frac{3\nu'(3-3\nu'-\mu')(9\nu'-1-\mu')}{(3\nu'-1)(3\nu'+1+\mu')(3\nu'+1-\mu')}$$

if $\qquad \mu' = +\sqrt{\{(3\nu'-1)^2-4\pi\nu'^3\}}$.]

From the above equation we may determine the value of ν for any value of μ, or *vice versâ*, and then at once obtain the corresponding values of the saturation pressure, temperature, and vapour volume π, τ, ν', as well as of ν'', so that we can easily plot the curves connecting these magnitudes with μ, or indeed with ν or τ, &c.

We easily find that $\mu=0$ for both $\nu=\frac{1}{3}$ and $\nu=1$ and that it has a maximum value ·3065658, to which correspond $\nu=\cdot5430962$, $\nu'=3\cdot365713$, $\pi=\cdot4713545$, $\tau=\cdot8371466$.

We further find, by somewhat laborious work, for the case of saturation

$$\frac{d\pi}{d\nu} = \mathrm{BFH}/\mathrm{A}\nu^4, \qquad \frac{d\nu'}{d\nu} = -\mathrm{EF}/\mu\mathrm{AB},$$

$$\frac{d\tau}{d\nu} = \mathrm{DEF}/8\nu^4;$$

and consequently

$$\frac{d\pi}{d\tau} = 8\mathrm{BH}/\mathrm{ADE}, \qquad \frac{d\nu'}{d\tau} = -8\nu^4/\mu\mathrm{ABD}.$$

The saturation specific heats

$$s \equiv \mathrm{K}+\mathrm{L}dp/dt, \qquad s' \equiv \mathrm{K}'+\mathrm{L}'dp/dt,$$

are thus given by

$$(\mathfrak{s}-\mathfrak{k})/R = (DE-BH)/FG = 3\nu/F$$

$$(\mathfrak{k}-\mathfrak{s}')/R = (H-AD)/\mu G \quad = 3\nu/\mu,$$

whence also

$$(\mathfrak{s}-\mathfrak{s}')/R = 3\nu G/\mu F.$$

Furthermore, the latent heat of vaporization λ is given by

$$\lambda/PV \equiv (\nu'-\nu)\tau\, d\pi/d\tau = GH/\nu^2\,;$$

the work of vaporization w by

$$w/PV \equiv \pi(\nu'-\nu) = CG/\nu^2\,;$$

and the real latent heat λ' by

$$\lambda'/PV \equiv \lambda/PV - \pi(\nu'-\nu) = 3G/\nu.$$

We may hence tabulate the values (p. 179) of, and plot curves for, all the above magnitudes with either ν, π, or τ for abscissæ (Pl. IV.), and we at once see that:

(i.) $\mathfrak{s}-\mathfrak{k}$ is always positive, increasing from R to ∞ as τ increases from 0 to 1:

(ii.) $\mathfrak{s}'-\mathfrak{k}$ is always negative and has a maximum value $-4\cdot95824\,R$ when $\mu = \cdot288953$, to which correspond $\nu = \cdot477565$, $\nu' = 6\cdot64469$, $\pi = \cdot238098$, $\tau = \cdot724335$; it is $-\infty$ when τ is either 0 or 1:

(iii.) inversion in the sign of $\mathfrak{s}'$ (which is negative for the highest and lowest temperatures) will therefore take place if $\mathfrak{k}/R > 4\cdot95824$, or, on the assumption that $\mathfrak{k}/R = N + \frac{1}{2}$ for an N-atomic gas, if $N > 4\cdot45824$, so that inversion will occur if the gas has at least *five* atoms in its molecule:

In van der Waals' lately published *Lehrbuch der Thermodynamik*, edited by Kohnstamm, it is found by an approximate calculation that for $\mathfrak{s}'$ to be positive $\kappa \equiv 1 + R/\mathfrak{k}$ must be less than 19/17, which is equivalent to $\mathfrak{k}/R > 8\cdot5$. Dalton (Phil. Mag. April 1907, p. 537) has given the condition $\kappa < 1\cdot202$ by an interpolation method, and the foregoing calculation gives $\kappa < 5\cdot95824/4\cdot95824 \equiv 1\cdot20168$.

(iv.) $\mathfrak{s}-\mathfrak{s}'$ is always positive and has a minimum value $7\cdot00789\,R$ when $\mu = \cdot262535$, which corresponds to $\nu = \cdot445963$, $\nu' = 11\cdot8364$, $\pi = \cdot127530$, $\tau = \cdot642488$:

Van der Waals' characteristic

μ.	ν.	ν'.	π.	τ.	$(\mathfrak{s}-\mathfrak{k})/R$.	$(\mathfrak{k}-\mathfrak{s}')/R$.	$\mathfrak{s}-\mathfrak{s}')/R$.	λ/PV.	$\pi(\nu'-\nu)$.	λ'/PV.
·0	·33333	∞	·0	·0	1·	∞	∞	9·	·0	9·
·05	·35	$5{\cdot}97\times10^{7}$	·00000	·15306	1·1053	21·	22·1053	8·9796	·4082	8·5714
·07399	·358	$1{\cdot}34\times10^{5}$	·00000	·2165	1·1598	14·5146	15·6744	8·9572	·5774	8·3799
·08524	·36176	$2{\cdot}59\times10^{4}$	·00002	·2443	1·1864	12·7317	13·9181	8·9442	·6515	8·2928
·12418	·375	911·12	·00097	·33334	1·2851	9·0597	10·3448	8·8823	·8856	7·9967
·15441	·38602	212·00	·00495	·39785	1·3725	7·5	8·8725	8·8041	1·0465	7·7575
·18368	·39797	77·759	·01533	·45950	1·4718	6·5	7·9718	8·6856	1·1859	7·4996
·18824	·4	68·026	·01784	·46920	1·4891	6·3748	7·8639	8·6620	1·2061	7·4559
·23013	·42191	23·706	·05875	·56175	1·6830	5·5	7·1830	8·3520	1·3680	6·9840
·26254	·44596	11·836	·12753	·64249	1·9119	5·0960	7·0079	7·9262	1·4526	6·4736
·26682	·45	10·820	·14076	·65432	1·9520	5·0596	7·0117	7·8491	1·4597	6·3894
·28199	·46724	7·8054	·20062	·70011	2·1298	4·9708	7·1006	7·5085	1·4722	6·0364
·28895	·47757	6·6447	·23810	·72433	2·2415	4·9582	7·1997	7·2988	1·4684	5·8304
·29955	·5	4·9888	·32054	·77003	2·4991	5·0075	7·5066	6·8375	1·4388	5·3987
·30306	·512	4·3960	·36405	·79115	2·6461	5·0683	7·7144	6·5909	1·4140	5·1769
·30657	·54310	3·3657	·47135	·83715	3·0622	5·3146	8·3768	5·9630	1·3305	4·6325
·30641	·55	3·2015	·49378	·84590	3·1622	5·3849	8·5471	5·8267	1·3093	4·5175
·30564	·56033	2·9856	·52627	·85816	3·3177	5·5	8·8177	5·6254	1·2763	4·3491
·29642	·60340	2·3488	·64700	·9	4·0524	6·1070	10·1593	4·8239	1·1293	3·6946
·29089	·61953	2·1826	·68609	·9125	4·3705	6·3894	10·7599	4·5402	1·0724	3·4678
·28864	·62538	2·1289	·69946	·91668	4·4926	6·5	10·9926	4·4396	1·0517	3·3879
·28345	·63785	2·0246	·72659	·925	4·7661	6·7509	11·5169	4·2291	1·0076	3·2215
·27353	·65903	1·8734	·76851	·9375	5·2766	7·2281	12·5047	3·8841	·9333	2·9508
·26795	·66988	1·8064	·78806	·94319	5·5636	7·5	13·0636	3·7133	·8956	2·8177
·26013	·68412	1·7271	·81188	·95	5·9705	7·8899	13·8604	3·4949	·8467	2·6481
·24156	·71494	1·5830	·85670	·9625	6·9909	8·8790	15·8699	3·0447	·7437	2·3010
·21437	·75514	1·4369	·90298	·975	8·7095	10·5675	19·2771	2·5006	·6156	1·8850
·16922	·81439	1·2785	·95075	·9875	12·6061	14·4378	27·0439	1·7786	·4413	1·3373
·0957	·9	1·1220	·98794	·99697	26·4317	28·2132	54·6449	·8788	·2193	·6595
·0	1·	1·	1·	1·	∞	∞	∞	·0	·0	·0

(v.) the work of vaporization has a maximum value $1{\cdot}47215\ \mathrm{PV} \equiv {\cdot}55206\ \mathrm{RT}$ when $\mu = {\cdot}281990$ (as also found by Ritter), to which correspond $\nu = {\cdot}467237$, $\nu' = 7{\cdot}80538$, $\pi = {\cdot}200617$, $\tau = {\cdot}700110$; it is 0 when τ is either 0 or 1:

(vi.) the greatest value of both λ and λ' is $9\ \mathrm{PV} \equiv 27\ \mathrm{RT}/8$, this occurring for $\mu = 0$, $\nu = \frac{1}{3}$, $\pi = 0$, $\tau = 0$, and both decrease continuously to the value 0 at the critical point.

Inversion of the sign of $\mathfrak{s}'$ occurs where $\mathfrak{s}' = 0$, that is, where $3\nu/\mu = \mathfrak{k}/\mathrm{R}$, and we can easily see, as was pointed out by Raveau* and earlier by Duhem, that this is at the points where the entropy along the vapour side of the connode has critical values.

On the assumption that $\mathfrak{k}/\mathrm{R} = \mathrm{N} + \frac{1}{2}$ we may therefore determine the temperatures of inversion for different values of N either graphically (i.) from the diagram of the values of $(\mathfrak{k} - \mathfrak{s}')/\mathrm{R}$ in terms of τ or (ii.) from the diagram giving the values of ν, τ, π in terms of μ, by finding the intersections of the lines $3\nu/\mu = \mathrm{N} + \frac{1}{2}$ with the $\nu\mu$-curve, or thus by calculation.

If we put $n \equiv 3(\mathrm{N} - \frac{1}{2})/(\mathrm{N} + \frac{1}{2})$, the desired points on the connode correspond to $\mu = (3 - n)\nu$ and are therefore given by

$$\log \frac{(6-n)\nu + 1}{(3\nu - 1)(n\nu - 1)} = \frac{3\nu(3 - n\nu)\{(12 - n)\nu - 1\}}{(3\nu - 1)(n\nu + 1)\{(6 - n)\nu + 1\}},$$

and for these we have

$$\pi = (n\nu - 1)\{(6 - n)\nu - 1\} / 4\nu^3$$

$$\tau = (3\nu - 1)(n\nu + 1)\{(6 - n)\nu + 1\} / 32\nu^3.$$

We also have for these points

$$\nu' = 2\nu/(3\nu - 1 - \mu) = 2\nu/(n\nu - 1),\ i.e.\ \nu = \nu'/(n\nu' - 2),$$

or

$$\log \frac{(3\nu' - 1)(n\nu' - 2)}{(3 - n)\nu' + 2} = \frac{3\nu'(n\nu' - 3)\{(6 - n)\nu' + 1\}}{(3\nu' - 1)(n\nu' - 1)\{(3 - n)\nu' + 2\}}$$

$$\pi = (n\nu' - 2)\{(3 - n)\nu' + 1\}/\nu'^3$$

$$\tau = (3\nu' - 1)(n\nu' - 1)\{(3 - n)\nu' + 2\}/8\nu'^3.$$

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* *Journ. de Phys.* 1892, p. 461.

We thus get for the points where inversion of s' occurs

N.	ν'.	π.	τ.	ν'.	π.	τ.
5	2·9855	·52627	·85816	23·706	·05875	·56175
6	2·1289	·69946	·91668	77·759	·01533	·45950
7	1·8064	·78806	·94319	212·00	·00495	·39785
8	1·63	·840	·958	543·88	·00172	·35353
9	1·52	·873	·967	1362·8	·000623	·31918
10	1·45	·899	·974	3389·4	·000229	·29136
11	1·40	·918	·979	8430·1	·0000848	·26818
12	1·34	·933	·983	21036·	·0000315	·24849

I have lately been interested to examine by this method the values of the heat-capacities, &c. of a saturated fluid in the neighbourhood of the critical point. For though v. d. Waals' equation is not a correct representation of the behaviour of gases, yet it represents so near an approximation that its indications are of value.

If we put $x \equiv 1-\nu$, then in the neighbourhood of the critical point μ and x are very small so that we may expand the terms in the equation of the connode in ascending powers of μ and x; we then obtain

$$0=\log\frac{4-3x+\mu}{(2-3x)(2-3x-\mu)}-\frac{3(1-x)(3x+\mu)(8-9x+\mu)}{(2-3x)(4-3x-\mu)(4-3x+\mu)}$$

$$=-\frac{9}{1024}(3x+\mu)^3\{x-\mu+\tfrac{1}{5}(17x^2-17x\mu-2\mu^2)+\tfrac{1}{4}(31x^3-31x^2\mu-7x\mu^2-\mu^3)$$

$$+\frac{1}{56}(837x^4-837x^3\mu-275x^2\mu^2-79x\mu^3-6\mu^4)+\dots\}$$

whence

$$x=\mu+\frac{2}{5}\mu^2+\frac{16}{25}\mu^3+\frac{229}{175}\mu^4+\dots$$

or

$$\mu=x-\frac{2}{5}x^2-\frac{8}{25}x^3-\frac{61}{175}x^4-\dots,$$

and thence we deduce

$$\tau=1-\frac{1}{4}x^2-\frac{9}{20}x^3-\frac{129}{200}x^4-\frac{5967}{7000}x^5-\dots.$$

If now we put $z^2 \equiv 1-\tau$, we get

$$\mu = 2z - \frac{26}{5}z^2 + \frac{227}{25}z^3 - \frac{13544}{875}z^4 + \dots$$

$$\nu = 1 - 2z + \frac{18}{5}z^2 - \frac{147}{25}z^3 + \frac{7992}{875}z^4 - \dots$$

$$\nu' = 1 + 2z + \frac{18}{5}z^2 + \frac{147}{25}z^3 + \frac{7992}{875}z^4 + \dots$$

$$\pi = 1 - 4z^2 + \frac{24}{5}z^4 - \frac{816}{875}z^6 + \dots$$

$$(\mathfrak{s} - \mathfrak{h})/\mathrm{R} = \frac{3}{2}z^{-1}\left(1 - \frac{3}{5}z + \frac{31}{50}z^2 - \frac{654}{875}z^3 + \dots\right)$$

$$(\mathfrak{h} - \mathfrak{s}')/\mathrm{R} = \frac{3}{2}z^{-1}\left(1 + \frac{3}{5}z + \frac{31}{50}z^2 + \frac{654}{875}z^3 + \dots\right)$$

$$\lambda/\mathrm{PV} = 16z\left(1 - \frac{23}{50}z^2 + \dots\right)$$

$$\pi(\nu' - \nu) = 4z\left(1 - \frac{53}{50}z^2 + \dots\right).$$

Thus in the neighbourhood of the critical point we may write $\pi = 4\tau - 3$ or, with greater accuracy,

$$\pi = 1 - \cdot 8(6\tau - 1)(1 - \tau),$$

and also $\lambda/\mathrm{RT} = 6\sqrt{(1-\tau)}$ or, with very great accuracy, $\lambda/\mathrm{RT} = \cdot 12(27 + 23\tau)\sqrt{(1-\tau)}$; further, the work of vaporization is one-quarter of the latent heat.

We likewise see that the mean of the saturation-densities near the critical point is $\cdot 1(11 - \pi)$ if $\cdot 04(1-\pi)^2$ is negligible, so that within this limit only the law of the straight diameter is exact with v. d. Waals' characteristic.

Saturation with Clausius' characteristic.

With Clausius' characteristic

$$p = \frac{\mathrm{R}t}{v - \alpha} - \frac{c}{t(v+\beta)^2}$$

the critical state is given by $\mathrm{V} = 3\alpha + 2\beta$, $\mathrm{P}^2 = c\mathrm{R}/216\gamma^3$, $\mathrm{T}^2 = 8c/27\gamma\mathrm{R}$, where $\gamma \equiv \alpha + \beta$; and if we write $\nu \equiv (v+\beta)/(\mathrm{V}+\beta)$, the *reduced* characteristic becomes

$$\pi = \frac{8\tau}{3\nu - 1} - \frac{3}{\tau\nu^2}.$$

In the case of *saturation* we obtain from Planck's memoir (*loc. cit.*)

$$\nu=\tfrac{1}{3}(r+1-r\cos\phi),\quad \nu'=\tfrac{1}{3}(r+1+r\cos\phi)$$

$$\tau=\frac{3\sqrt{3}(r+1)^{\frac{1}{2}}\,r\sin\phi}{2(r^2\sin^2\phi+2r+1)}$$

$$\pi=\frac{6\sqrt{3}(r^2\sin^2\phi-1)}{(r+1)^{\frac{3}{2}}(r^2\sin^2\phi+2r+1)r\sin\phi}$$

$$\frac{\lambda/\tau}{\mathrm{RT}}=\frac{3r^2\sin^2\phi+4r+1}{(r+1)\,r\sin\phi\tan\phi},$$

where

$$r+1\equiv\frac{\cos\phi\cot^2\phi}{\log\cot\frac{1}{2}\phi-\cos\phi};$$

our previous mode of calculation then gives for the saturation specific heats

$$\frac{\mathfrak{s}-\mathfrak{k}}{\mathrm{R}}\text{ and }\frac{\mathfrak{k}-\mathfrak{s}'}{\mathrm{R}}=\frac{2(r^2\sin^2\phi+2r+1)}{r^2\sin^2\phi}$$

$$\times\frac{(r^2\sin^2\phi-1)^2-3(r+1)\pm(2r^2\sin^2\phi-1)r\cos\phi}{(r^2\sin^2\phi-1)(2r+2-r^2\sin^2\phi)-2(r+1)},$$

and the work of vaporization w is $\frac{1}{4}\mathrm{RT}\pi r\cos\phi$.

Calculation for different values of ϕ gives the following table (p. 184), whence it appears that

(i.) $\mathfrak{s}-\mathfrak{k}$ is always positive, is infinite for $\tau=0$ and $\tau=1$, and has a minimum value 15·333 R for $\tau=\cdot83235$:

(ii.) $\mathfrak{s}'-\mathfrak{k}$ is always negative, is $-\infty$ for $\tau=0$ and $\tau=1$, and has a maximum value $-11\cdot355$ R for $\tau=\cdot80579$:

(iii.) $\mathfrak{s}-\mathfrak{s}'$ is always positive, is ∞ for $\tau=0$ and $\tau=1$, and has a minimum value 26·741 R for $\tau=\cdot81962$:

(iv.) the latent heat λ increases continuously from 0 to ∞ as τ falls from 1 to 0:

(v.) the work of vaporization has a maximum value ·68567 RT when $\tau=\cdot76610$, being 0 for $\tau=0$ and $\tau=1$.

With this characteristic too inversion may occur in the sign of $\mathfrak{s}'$, but now when $\kappa<1\cdot088$ or, on the former assumption that $\mathfrak{k}/\mathrm{R}=\mathrm{N}+\frac{1}{2}$ for an N-atomic gas, only if the gas has at least *eleven* atoms in its molecule.

Clausius' characteristic.

ϕ.	r.	τ.	π.	$(\mathfrak{s}-\mathfrak{h})/R.$	$(\mathfrak{h}-\mathfrak{s}')/R.$	$(\mathfrak{s}-\mathfrak{s}')/R.$	$\lambda/RT.$	$w/RT.$
0°	∞	·0	·0	∞	∞	∞	∞	·0
0° 2′	413648	·45426	·00000	35·173	30·570	65·743	14·341	·45426
0° 30′	2959·9	·55428	·00075	24·761	20·060	44·821	11·496	·55324
1°	876·09	·59209	·00269	21·961	17·033	38·994	10·640	·58880
2°	267·81	·63805	·00939	19·799	14·844	34·643	9·7036	·62801
4°	85·551	·69443	·03119	17·632	12·858	30·490	8·6528	·66551
6°	45·069	·73302	·06078	16·571	12·002	28·573	7·9648	·68105
8°	29·021	·76299	·09543	15·964	11·586	27·550	7·4335	·68562
8° 14′	27·792	·76610	·09971	15·912	11·556	27·467	7·3781	·68567
10°	20·823	·78833	·13332	15·609	11·402	27·010	6·9917	·68349
11° 43½′	16·526	·80579	·16761	15·435	11·355	26·790	6·6584	·67798
12°	15·982	·80844	·17318	15·416	11·362	26·778	6·6090	·67681
13° 12′	13·958	·81962	·19761	15·358	11·384	26·741	6·3995	·67132
14°	12·849	·82656	·21403	15·338	11·416	26·755	6·2673	·66709
14° 42′	12·004	·83235	·22844	15·333	11·453	26·787	6·1556	·66310
16°	10·679	·84246	·25759	15·349	11·545	26·894	5·9575	·65503
18°	9·1029	·85659	·29628	15·432	11·733	27·164	5·6726	·64126
20°	7·9144	·86924	·33681	15·575	11·969	27·544	5·4078	·62622
30°	4·7616	·91698	·52431	17·032	13·775	30·807	4·2900	·54051
45°	3·0576	·95973	·74371	21·591	18 630	40·222	2·9816	·40199
60°	2·3802	·98365	·89014	31·450	28·646	60·096	1·8966	·26483
80°	2·0371	·99830	·98811	91·588	88·836	180·424	·61312	·08739
90°	2·	1·	1·	∞	∞	∞	·0	·0

The contrast between these conditions and the corresponding ones for v. d. Waals' characteristic, especially in regard to $\mathfrak{s}-\mathfrak{k}$ and λ, is very marked and is very clearly shown by the curves that have been plotted. This contrast subsists further in the neighbourhood of the critical point; for, if as before we write $z^2 \equiv 1-\tau$, we find $\pi = 1-7z^2$, $\lambda/\mathrm{RT} = (21/\sqrt{2})z = 14{\cdot}849z$, $(\mathfrak{s}-\mathfrak{k})/\mathrm{R} - 1{\cdot}35 = 7{\cdot}425/z = (\mathfrak{k}-\mathfrak{s}')/\mathrm{R} + 1{\cdot}35$, and the work of vaporization is one-seventh of the latent heat.

In this paper the Author has shown that by use of a special variable exact expressions may be found with van der Waals' characteristic for the specific heats $\mathfrak{s}, \mathfrak{s}'$ of saturated liquid and vapour and for all other magnitudes connected with the state of saturation; and it is deduced that, if $\mathfrak{k}$ denotes the isometric specific heat, which is constant or a function of the temperature only, (i) $\mathfrak{s}-\mathfrak{k}$ is always positive, increasing from R to ∞ as the reduced temperature τ rises from 0 to 1; (ii) $\mathfrak{k}-\mathfrak{s}'$ is always positive, having a minimum value $4{\cdot}96$ R when $\tau = {\cdot}72$ and being ∞ when τ is either 0 or 1; (iii) inversion in the sign of $\mathfrak{s}'$ can thus occur when $\mathfrak{k}/\mathrm{R} > 4{\cdot}96$ or

$$\kappa = 1 + \mathrm{R}/\mathfrak{k} < 1.202,$$

so that, on the assumption that

$$\mathfrak{k}/\mathrm{R} = \mathrm{N} + \tfrac{1}{2}$$

for an N-atomic gas, inversion can occur only if the gas has at least *five* atoms in its molecule; (iv) the latent heat of vaporization increases continuously from 0 to its largest value $(27/8)\mathrm{RT}$ as τ falls from 1 to 0, T being the absolute critical temperature: (v) the work of vaporization has a maximum value ${\cdot}55$ RT when $\tau = {\cdot}70$.

Clausius' characteristic similarly treated gives widely different results: (i) while $\mathfrak{s}-\mathfrak{k}$ is always positive, it is ∞ when τ is either 0 or 1, having a minimum value $15{\cdot}3$ R for $\tau = {\cdot}83$; (ii) $\mathfrak{k}-\mathfrak{s}'$ is always positive, is ∞ when τ is either 0 or 1 and has a minimum value $11{\cdot}36$ R for $\tau = {\cdot}81$; (iii) inversion in the sign of $\mathfrak{s}'$ can thus only occur if

$$\mathfrak{k}/\mathrm{R} > 11{\cdot}36 \quad \text{or} \quad \kappa < 1{\cdot}081,$$

or, on the above assumption, if there are at least *eleven* atoms in the molecule; (iv) the latent heat of vaporization increases continuously from 0 to ∞ as τ falls from 1 to 0; (v) the work of vaporization has a maximum value ${\cdot}69$ RT when $\tau = {\cdot}77$.

The contrast between these results is especially marked for $\mathfrak{s}-\mathfrak{k}$ and the latent heat; it subsists further in the neighbourhood of the critical point where $d\pi/d\tau$ is 4 (v. d. W.) and 7 (Cl.), $\lambda/\mathrm{RT}\sqrt{(1-\tau)}$ is 6 (v. d. W.) and $14{\cdot}8$ (Cl), $(\mathfrak{s}-\mathfrak{k})\sqrt{(1-\tau)}/\mathrm{R}$ is $1{\cdot}5$ (v. d. W.) and $7{\cdot}4$ (Cl.).

DISCUSSION.

Prof. C. H. **Lees** expressed his interest in the paper and remarked on the utility of the introduction of the new variable μ. He asked if a simple relation could be given between saturation-pressure and temperature in terms of this variable. The very different results the Author had obtained from van der Waals' and from Clausius' equations might suggest a crucial experiment which would decide in favour of one of them.

The **Author**, in reply to Prof. Lees, said he had not been able by means of his special variable to express in simple terms the relation between saturated pressure and absolute temperature.

XIV. *The Polarization of Dielectrics in a Steady Field of Force. By* W. M. Thornton, *D.Sc., D.Eng., Professor of Electrical Engineering in Armstrong College, Newcastle-on-Tyne*[*].

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1. INTRODUCTION.

THE slow polarization of a dielectric ellipsoid suspended in a steady field of force can be followed by observing from time to time the period of small swings about the line of the field[†]. In a perfect insulator the polarization reaches its full value in a much shorter time than the period of such a moving system, and a single observation of the latter would suffice to determine the dielectric constant of the ellipsoid. In all solid dielectrics there is, however, slow electrical movement under the influence of the field within the ellipsoid and a consequent accumulation of charge at each end. The restoring couple, and therefore the apparent dielectric constant, increase until the limit to molecular polarization is reached.

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[*] Read January 21, 1910.

[†] Roy. Soc. Proc. A. vol. lxxxii. p. 422 *et seq.*

The advantage of the method is that the "conduction" current is confined to the separation of charge within the solid symmetrically about the central section.

An account is given below of observations made upon the behaviour of various solid dielectrics, in which the total polarization is expressed in terms of the apparent dielectric constant. From these observations the resistivity of insulators has been found in a new way.

A difference of one part in a thousand in the dielectric constant can be readily measured by the suspended ellipsoid method. On account of this sensitiveness three stages in the polarization were found to be present in all the homogeneous substances examined. First a sudden very rapid yield, then a slow change at a uniform rate, and finally an approach to saturation. The first suggests a normal slackness of charge in the dielectric molecule, the second is probably a continuation of the polarization by the influence of adjacent molecular charges, a slow linear change of the nature of a simple strain. In the third stage the rate of displacement ceases to be uniform, and the polarization reaches a maximum which is shown in the paper to be independent of the intensity of the polarizing field within the specimen.

According to this "three stage" theory, the quantity passing in the instantaneous charge or discharge of a condenser with a solid dielectric, involving as it does the first stage only, should be proportional to the voltage; the total residual *charge* corresponding to the later stages, the apparent dielectric constant of which is independent of voltage, should also be proportional to the voltage, as shown by Hopkinson* to be the case. Dielectric hysteresis loss should decrease with rise of frequency, on account of the smaller range possible in the second stage, which agrees with the experimental results of Threlfall †.

It was found further that there was no permanent residual electrification in the dielectrics examined, showing that the polarization, however long continued and slow in action, was quasi-elastic in type.

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* Roy. Soc. Phil. Trans. vol. clxvii. part ii. pp. 599-626 (1877).

† R. Threlfall, "On the Conversion of Electric Energy in Dielectrics," part iii. Physical Review, vol. v. p. 65.

When testing electric cables the charging current is sometimes measured after one minute's electrification. With the usual insulators this is only in part a true conduction current, the rest being the second stage of polarization. What the latter would be in any condenser can be calculated from the curves of rate of change of apparent dielectric constant given later, and in this way the true conductivity found.

2. Experiments on Quartz.

The first systematic examination of the influence of time of charge upon the values obtained for dielectric constants was made by Boltzmann[*], and by Romich and Nowak[†] under his direction. They obtained for quartz, for example, a value greater than 1000 in steady fields. All subsequent observers recognizing these results have specified the duration of charge. The effect was dealt with by J. Hopkinson[‡] in an important series of papers, and the recent work of Beaulard[§] upon the dielectric constant of water at high frequencies is a good illustration of it. With the exception of Romich and Nowak's experiments, observations of the influence of steady fields have been mostly made on the charge and discharge currents of condensers, and in their interpretation the relative importance of polarization and conduction must be separated. By a conduction current is meant one produced by the passage of free electrons or ions, and this takes place much more freely when the dielectric is in direct contact with the terminal plates between which the pressure is maintained. A very complete *resumé* of work on dielectrics up to 1904 is given by L. Graetz in Band iv. of Winkelmann's *Handbuch der Physik*. A paper by Von Schweidler, in *Annalen der Physik*[‖], deals with more recent work and gives a full bibliography[¶].

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[*] *Wien. Ber.* lxvi., lxvii. (1872-3).

[†] *Wien. Ber.* lxx. (1874).

[‡] Original Papers, vol. ii. nos. 18, 19, 27.

[§] Science Abstracts, 1905 to 1908.

[‖] *Ann. der Phys.* xxiv. p. 711 (1907).

[¶] See also H. A. Wilson, Roy. Soc. Proc. A. lxxxi. p. 409, "On Electrostatic Induction through Solid Insulators."

In the present experiments the first substance examined was quartz cut parallel to the optic axis and formed into an ellipsoid 2 cm. long and ·2 cm. diameter. It was very carefully cleaned, dried, and suspended by a quartz fibre about ·001 cm. diameter and 20 cm. long between flat brass plates 6 cm. apart, contained in a vessel dried under vacuum and by phosphoric anhydride for several weeks. The leak between the plates would therefore be exceedingly small, and with a potential-difference of 480 volts between the plates the observed periods were the same in dry air and in vacuo. In order to obtain more than two figure accuracy by this method it is necessary to take the mean of at least five observations of the period. In the first few minutes this was as a rule changing quickly, and it was only in the later stages of the polarization that several readings could be taken.

Particulars of the form and dimensions of the specimens used in this paper are given in that dealing with alternating fields*.

The free period of the quartz ellipsoid was 63·3 seconds. The following table gives the observed periods in fields of various strength for the first 20 minutes of polarization.

Table I.

F is the intensity of the external polarizing field in Electrostatic Units.

F.	10 s.	15 s.	30 s.	1 m.	2 m.	3 m.	4 m.	5 m.	6 m.	7 m.	8 m.	10 m.	15 m.	20 m.
·087	...	51·5	46·5	42·7	40·0	38·0	36·6	35·5	34·5	34·0	33·6	33·0	31·8	31·3
·116	...	40·0	36·5	32·7	30·0	28·7	28·0	27·2	26·6	26·0	25·6	25·2	24·0	23·6
·179	...	35·0	30·0	24·5	21·5	20·5	20·0	19·5	19·0	18·7	18·6	18·1	17·0	16·0
·284	32·0	...	21·5	18·0	16·0	15·0	14·2	13·5	13·2	13·0	12·5	11·7	10·5	10·2
·404	21·1	...	15·0	13·0	11·6	10·7	10·5	9·7	9·2	9·0	8·7	8·2	7·5	7·2
·506	15·1	...	13·0	11·3	10·0	9·0	8·5	8·0	7·5	7·2	7·0	6·7	6·2	6·0
·747	11·6	...	10·0	8·75	7·3	6·5	5·75	5·35	5·0	4·75	4·73	4·35	4·05	4·0

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* Roy. Soc. Proc. *loc. cit.*

The time at the head of each column is that at which the observation was made; the figures in the column are the measured period in seconds. After each set of readings the ellipsoid was allowed to stand about twelve hours to depolarize. On reversing the polarity of the plates after about one minute's electrification the ellipsoid turned through 180 degrees and followed any subsequent reversal. In five minutes the quartz was sufficiently electrified to pick up pieces of tissue-paper one millimetre square.

Later observations showed that 20 minutes was not long enough to establish full polarization. The ellipsoid was therefore resuspended, having now a free period of 34·73 seconds. After 120 hours in a field of 0·137 electrostatic unit the period was 17 seconds; after 24 hours longer in a field increased to ·266 unit, 9·72 seconds.

Values of the dielectric constant calculated from the restoring couple on an ellipsoid making small swings in an alternating field of force are in close agreement with the best previous determinations. The assumption made in calculating the couple is that the polarization is taken up instantaneously in any position, and the agreement shows this to be justified. When, however, the field is unidirectional the components of polarization are not the same along and at right angles to the axis of the ellipsoid. It will be shown subsequently that several days are, in general, required to reach a steady state when the ellipsoid is allowed to stand in line with the field. There is in this position no transverse polarization, but if the ellipsoid is then given small swings there will be one which alternates with the motion and may thus be considered to be proportional to the constant obtained in alternating fields.

The ellipsoid behaves in fact as if it were in effect crystalline, and the restoring couple at unit angular displacement is

$$\left(\frac{\kappa_1}{1+\kappa_1 N}-\frac{\kappa_2}{1+\kappa_2 L}\right)VF^2. \quad . \quad . \quad . \quad . \quad (1)$$

Where κ_1 is the longitudinal and κ_2 the transverse susceptibility, N and L the longitudinal and transverse reaction coefficients, V the volume of the ellipsoid, and F the intensity of the external field.

This couple is equal to $4\pi^2 I(n^2-n_0^2)$, where I is the moment of inertia of the suspended system; n, n_0, the frequency of swing with and without the field.

Hence

$$\left(\frac{\kappa_1}{1+\kappa_1 N}-\frac{\kappa_2}{1+\kappa_2 L}\right)VF^2=4\pi^2 I\,(n^2-n_0^2). \quad . \quad (2)$$

Writing

$$\frac{4\pi^2 I(n^2-n_0^2)}{VF^2}+\frac{\kappa_2}{1+\kappa_2 L}=a,$$

we have

$$\kappa_1=a/(1-Na),$$

and the dielectric constant

$$K_1=\frac{1+(4\pi-N)a}{1-Na}.$$

For a *quartz ellipsoid cut parallel to the optic axis* $L=6{\cdot}133$, $N={\cdot}3013$, $V={\cdot}0534$ c. c., $I={\cdot}02912$, $K_0=4{\cdot}60$.

From the later set of observations given above $n^2={\cdot}0106$ in a field ${\cdot}266$, $n_0^2={\cdot}00083$. Thus $F^2/(n^2-n_0^2)=7{\cdot}25$ and $a=3{\cdot}069$, giving finally $\mathbf{K=508}$. (Quartz $\parallel$).

In the weaker field, $F={\cdot}137$, $F^2/(n^2-n_0^2)=7{\cdot}16$, giving $\mathbf{K=610}$.

An *ellipsoid of quartz cut perpendicular to the axis* had $L=6{\cdot}125$, $N={\cdot}313$, $V={\cdot}056$, $I={\cdot}0307$, $K_0=4{\cdot}548$.

Its free period was 33 seconds. After exposure to a field 0·26 for 48 hours the period was 19·3 seconds; after 41 hours longer in a field 0·132, 26·75 seconds.

From these we have the steady values

$$\begin{aligned} F^2/(n^2-n_0^2)&=38{\cdot}5,\ F=0{\cdot}26,\\ \text{,,}\qquad &=36{\cdot}4,\ F=0{\cdot}132. \end{aligned}$$

$a={\cdot}6705$, and $\mathbf{K=11{\cdot}66}$. (Quartz $\perp$).

Thus both in alternating and direct fields quartz $\parallel$ has the larger values, but whereas in the former the ratio of the dielectric constants is 1·012, in the latter it is about 45.

For the purpose of section 6 infra, it is necessary to know not only the final values but the rate of growth of the polarization. The following observations show the exceedingly slow rate of polarization of quartz $\perp$ compared with quartz $\parallel$.

The intensity of the field in all the cases given in the paper not otherwise stated was about ·265 electrostatic unit.

Quartz ⊥. $K \sim\, = 4{\cdot}548$. Free period $T_0 = 11{\cdot}82$ s.

Time	15 s.	35 s.	50 s.	16 m.	120 m.	48 h.
Period	11·2	11·05	11·02	11·0	10·57	—*
K	4·73	5·2	5·69	7·55	10·92	11·66

Quartz ‖. $K \sim\, = 4{\cdot}60$. $T_0 = 63{\cdot}3$ s.

Time	1 m.	2	4	6	10	15 m.	120 h.
Period	17·2	15·8	14·6	13·2	12·0	11·4	—*
K	20	27·3	39	53	78	110	610

* On another suspension.

From the readings of Table I., we have now for the latter ellipsoid after 20 minutes' electrification.

F	·087	·116	·129	·284	·404	·506	·747
$F^2/(n^2 - n_0^2)$	8·8	8·8	8·8	8·26	8·8	9·3	8·92

The mean of $F^2/(n^2 - n_0^2)$ is 8·81, and since the derived value of K depends upon this we have the remarkable fact that the polarization approaches the same value irrespective of the intensity of the polarizing field. This suggests that the slow polarization is a consequence of internal molecular attractions started by the initial polarization and maintained to a limit by the continuance of the field, but not in any way proportional to it. This is considered further in section 7.

Fused quartz has been largely employed in the construction of electroscopes for use in measurements of radioactivity. It is conceivable, though not highly probable, that some of the earlier observed feeble radioactive effects may have been in some measure due to the slow absorption of charge by the quartz support, for the rate of charge or discharge of such an instrument must be influenced by the polarization of the support.

3. Experiments on Flint Glass.

Similar measurements to those on quartz were made with flint glass specimens.

Cylinder of density 4·6.—Free period 23·8 seconds. After 5 days in $F = \cdot 132$, period 19·5 s.; after 48 hours longer in $F = \cdot 272$, period 13·8. From these $F^2/(n^2 - n_0^2) = 21{\cdot}86$ and 20·25 respectively, giving **K=580.**

Ellipsoid of density 4·1.—Free period 41·5 seconds. After 48 hours in $F = \cdot 132$, period 23·75; after 48 hours longer in $F = \cdot 272$, period 12·5. Here $F^2/(n^2 - n_0^2) = 14{\cdot}1$ in the lower field, 12·4 in the higher. The corresponding mean value of **K is 520.** On a fresh suspension the following rate of polarization was observed.

$$K_{\sim} = 8{\cdot}52.$$

Time	30 s.	2·5 m.	4·5 m.	6·5 m.	19 h.
K	27·9	37·7	47·0	50	200

Ellipsoid of density 3·3.—Free period 14·9 seconds. After 22 hours in $F = \cdot 271$, period 12·33; after 48 hours longer in $F = \cdot 136$, period 13·9. For the former $F^2/(n^2 - n_0^2) = 26{\cdot}85$, and the latter 27·2.

$$K_{\sim} = 6{\cdot}98.$$

Time	1 m.	15 m.	1 h.	2 h.	4 h.	10 h.	22 h.
K	7·0	8·1	10·5	12·4	16	20·6	22·6

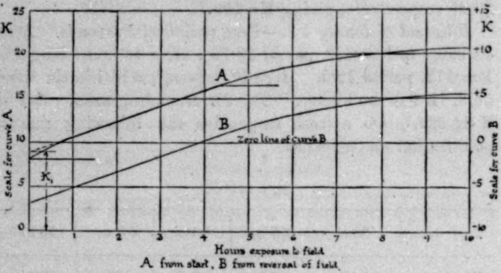
On reversing the field the ellipsoid did not reverse. The rate of fall of K under a reversing field and the early part of its rise after passing through zero are given below.

Field reversed.

Time	12 m.	90 m.	230 m.	5 h.	7 h.	9 h.	11 h.
K	−6·5	−4·0	0	1·6	3·7	4·8	5·6

This shows the *forced depolarization* of a dielectric by a method distinct from discharge through a galvanometer. These rates of change are drawn in fig. 1. Their chief interest lies in the fact that whether the polarization is increasing from zero or being reversed, the rate of change is for a long time the same. It is also noticeable that the time occupied in the linear polarization is nearly the same in both cases. The value of K before the field was reversed was 25, of which 9 was the sudden rise at the start. At reversal there was a rapid change equal to twice that from zero, *i. e*, of 18, leaving a polarization of K = 7 in a direction opposite to that of the field. It will be seen that Curve B starts from a value −7.

Fig. 1.



The ellipsoid was then exposed to the field for three weeks and acquired a value of K = 6075.

The following numbers give the *free depolarization* from this high value. The periods from which K was calculated were taken by applying a weak field and measuring the period after it had been on for a minute in each case, after which it was cut off. There is no doubt a sudden rise of polarization at the moment of applying the field, but since the conditions are the same in each case the figures give an approximation to the rate of relaxation.

Free depolarization of Flint Glass, Δ 3·3.

Time	10 m.	1 h.	2 h.	3 h.	4 h.	5 h.	19·5 h.
K	6075	426	79·7	28·0	12·73	9·0	8·17

4. Equality of Final Values.

Before proceeding to examine whether these high constants are true polarization as distinct from the movement of free charge, four typical substances were examined for equality of K in different fields, the periods being taken to be steady when they did not change by a measurable amount in twelve hours. Since the derived value of K is proportional to $F^2/(n^2-n_0^2)$ equality in observed values of the latter is also true for the dielectric constants.

Paraffin Wax Ellipsoid.—Free period 46·5 seconds. After 7 days in $F=\cdot259$, period 13·63; after 48 hours more in a field ·132, 24·0 seconds. From these $F^2/(n^2-n_0^2)=$ **13·65** and **13·69** respectively, giving equality of K.

Sulphur Cylinder.—Free period 23·1. After 16 hours in $F=\cdot131$, period 19·7; after 24 hours longer in $F=\cdot259$, 10·8 seconds. $F^2/(n^2-n_0^2)=$ **24·62** and **25·0** respectively.

Gutta Percha Cylinder.—Free period 49·6 seconds. After 66 hours in a field ·26, period 9·01; after 30 hours longer in $F=\cdot132$, 16·7 seconds. $F^2/(n^2-n_0^2)=$ **5·66** and **5·50.**

Ebonite Cylinder.—Free period 86·2 seconds. After 48 hours in $F=\cdot26$, period 12·64; after 48 hours longer in $F=\cdot13$, 24·0 seconds. $F^2/(n^2-n_0^2)=$ **11·1** and **10·85.**

Thus for different substances and over a wide range of values the effective polarization of an insulated rod approaches the same final value whatever the intensity of the field may be.

5. Rate of Polarization.

In the foregoing observations of the rate of change of K in steady fields there appear then to be three stages, a sudden yield, followed by a change at a uniform rate for intervals of time differing in the various substances from a few minutes to several hours, and later an approach to saturation.

In order to see how these, especially the second, hold over a wider range of substances the following measurements were made. The details of the observations and their reduction, which are lengthy, are not given. In every case the specimen was dried for several days, in some cases weeks, over phosphoric anhydride and *in vacuo*, until the period tested in an alternating field of force had become steady.

Fused Quartz, $K_{\sim}=3{\cdot}78$.

Time	1 m.	4 m.	7 m.	16 m.	30 m.	120 m.
K	6·8	8·3	9·7	12·8	14·7	14·7

Paraffin Wax, $K_{\sim}=2{\cdot}32$.

Time	·7 m.	1 m.	2 m.	5 m.	10 m.
K	6·28	6·52	6·77	7·04	7·12

Sulphur, $K_{\sim}=4{\cdot}03$.

Time	·2 m.	1 m.	2 m.	6 m.	10 m.	50 m.	2·5 h.	19 h.	27 h.
K	4·4	4·51	4·59	4·99	5·43	10·67	20·8	101	141

India Rubber, $K=3{\cdot}08$.

Time	1 m.	2 m.	4·5 m.	12·5 m.	32 m.	60 m.
K	5·0	5·78	7·66	11·8	18·6	24·5

Ebonite, K∼=2·79.

Time	1 m.	2 m.	4 m.	7 m.	10 m.
K	6·25	8·0	11·8	16·9	21·8

Resin, K∼=3·09.

Time	·5 m.	2 m.	4 m.	6 m	10 m.
K	7·3	7·86	8·19	8·47	8·9

Amber, K∼=2·8.

Time	1 m.	2 m.	3 m.	5 m.	11 m.	15 m.
K	8·0	13·1	19·1	32	54·2	61

Canada Balsam, K∼=2·72.

Time	·3 m.	1 m.	2 m.	4 m.	6 m.	10 m.
K	20·8	23·5	27·7	34·8	41·7	53

Gutta Percha, K∼=4·43.

Time	·8 m.	1·6 m.	3 m.	7 m.	10 m.
K	32	61·2	107	194	240

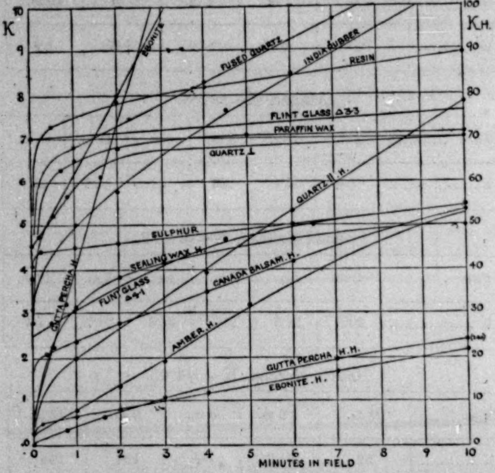
Sealing-Wax, K∼=4·56.

Time	·5 m.	1 m.	2 m.	3 m.	6 m.	10 m.
K	22·5	31·2	38·1	41·5	46·8	50

It will be noticed that gutta-percha, upon which so much depends in the working of submarine cables, has the highest values of all the substances examined. It is at least possible that some of the difficulties of cable working arise from this cause. If by mixing a suitable insulator of lower dielectric constant with it, or by some system of grading, retaining at the same time the impermeability to water, the effective constant could be reduced, the rate of signalling might be accelerated.

Curves of the above rates of change are drawn in fig. 2

Fig. 2.—Rate of Polarization of Dielectrics in Steady Fields.



for the first ten minutes of electrification, and it will be seen that the features of a rapid rise at the start and a straight line change following it are present in all but sealing-wax, a mechanical mixture. The linear change in most cases lasts for several hours. In the case of fused quartz the polarization is steady in half an hour, and no increase was observed after several days in the field. This rapid polarization is possibly the reason for its use as an insulator in electroscopes.

The initial rush is in almost every case over within a minute. This may explain why observations of cable resistance are taken after this interval.

6. MEASUREMENT OF SPECIFIC RESISTANCE.

The fact that the rate of change of the effective dielectric constant is uniform for some time after the field is applied, provides a method of measuring the specific resistance of a dielectric during polarization as distinct from that using the "time of relaxation."

Whatever the nature of the electrical movement during the polarization of an insulated ellipsoid may be it is in effect a current, the density of which is the rate of change of the total polarization in unit volume. The field within the ellipsoid is given by $F_1 = F/(1+\kappa N)$, where $\kappa = (K-1)/4\pi$. The polarization is then KF_1, and writing the apparent specific resistance ρ, we have

$$\frac{d}{dt}(KF_1) = \frac{F_1}{\rho}. \quad \ldots\ldots \quad (3)$$

The resistance measured in this way differs, though not to a marked extent, from that found by the passage of a steady current through the dielectric.

It may be said that in view of the facts of slow polarization the measurement of the current after one minute's electrification is meaningless as an indication of the *resistance* of the insulation. A large part of the current is polarization change. The leakage method of measuring the resistance of insulators is also complicated by this slow depolarization of the dielectric and greatly influenced by the time of charge.

We have from the above

$$\frac{\rho}{4\pi}\frac{dK}{dt} = 1 + \left(\frac{K-1}{4\pi}\right)N, \quad \ldots\ldots \quad (4)$$

from which ρ may be found at any stage of the polarization *.

Since the measurements are in electrostatic units, ρ is also, and in electromagnetic measure is

$$= v^2\{4\pi + (K-1)N\}/dK/dt \,.\, 10^9 \text{ ohms per cm. cube}, \quad . \quad (5)$$

where v is the ratio of the electromagnetic to the electrostatic unit of charge.

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* The statement of this section has been modified at the suggestion of Dr. T. H. Havelock, to whom the author is indebted for helpful criticism throughout the work.

The following table gives values of the specific resistance found in this way from the observed rates of change of fig. 2. The indiarubber and gutta-percha were not specially prepared for cable work.

The ratio of Curie's values for quartz ⊥ and || is 240; those now obtained give 130. It is interesting to note that the velocity of light in quartz is greatest when the wave-front and current are in the direction of least electrical conductivity. In accordance with previous observations the lighter flint-glass has the higher resistivity. Paraffin-wax was the most perfect insulator examined.

Table II.

Specific Resistance of Dielectrics at 17° C.

Substance.	dK/dt.	ρ ohms per cm. cub.		Temp.
Paraffin-Wax	·00026	$4·9 \times 10^{16}$	$3·4 \times 10^{16}$ Ayrton and Perry.	46
Quartz ⊥	·00064	$2·0 \times 10^{16}$	$·91 \times 10^{16}$ Curie.	
Flint-Glass Δ 3·3 ...	·00131	$9·9 \times 10^{15}$		
Sulphur	·00163	$8·2 \times 10^{15}$	c. 10×10^{15} Threlfall.	
Resin	·00197	$7·0 \times 10^{15}$		
Fused Quartz	·00805	$1·6 \times 10^{15}$	$3·8 \times 10^{11}$ Exner.	
Indiarubber	·0089	$1·5 \times 10^{15}$	$1·5 \times 10^{15}$ Electrician.	
Ebonite	·0316	$4·5 \times 10^{14}$	$2·8 \times 16^{16}$ Ayrton and Perry.	46
Canada Balsam......	·059	$2·8 \times 10^{14}$		
Flint-Glass Δ 4·1 ...	·077	$2·5 \times 10^{14}$		
Amber	·097	$1·55 \times 10^{14}$		
Quartz ||	·107	$1·53 \times 10^{14}$	$·38 \times 10^{14}$ Curie.	
Gutta-percha	·35	$6·21 \times 10^{13}$	$4·5 \times 10^{14}$ Lat. Clark. $2·5 \times 10^{13}$ F. Jenkin.	

It follows from equation (5) that when the period has reached a steady state the apparent resistance of the dielectric is infinite. This does not mean that no current could pass through it if it were in contact with the poles but that the polarization has reached its limit.

Although the dielectric constant increases at a uniform rate the apparent resistance increases also, as calculated below, from the curves of fig. 2. This indicates that, as one would expect, the mutual attraction of charges on adjacent molecules increases in intensity as they approach, so that though the resistance to movement increases, the rate, on the whole, is maintained uniform.

Change of Apparent Resistivity with Time.

Light Flint-Glass, Δ 3·3.

Time ...	1 m.	15 m.	1 h.	2 h.	4 h.	10 h.	15 h.	22 h.
ρ	·987 $\times 10^{16}$	1·56	2·0	2·7	3·65	10·28	40·4 $\times 10^{16}$	15·15 $\times 10^{18}$

Sulphur.

Time ...	·2 m.	2 m.	6 m.	10 m.	50 m.	2·5 h.	19 h.	27 h.
ρ	·816 $\times 10^{16}$	·818	·83	84	1·06	1·39	3·03	3·54 $\times 10^{16}$

Indiarubber.

Time ...	·5 m.	1 m.	2 m.	4·5 m.	12·5 m.	32 m.	60 m.
ρ	·057 $\times 10^{16}$	·084	·105	·136	·29	·72	1·66 $\times 10^{16}$

Fused Quartz.

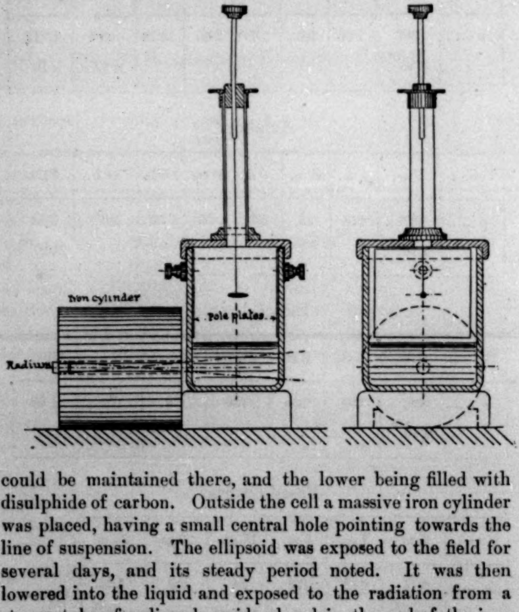
Time ...	·5 m.	1 m.	5 m.	15 m.	20 m.	25 m.
ρ	·071 $\times 10^{16}$	·112	·179	·24	·39	1·15 $\times 10^{16}$

7. NATURE OF CURRENT DURING SLOW POLARIZATION.

The electrical movement within the substance during slow polarization may be a translation of free electrons, as in metallic conduction, a transfer of charge, as in electrolysis, or a separation of the internal charges of the molecule, so that its electrical moment increases.

In order to determine which of these occurs in the present case, a flint-glass ellipsoid was suspended in a glass cell (fig. 3), the upper part having pole plates, so that a field could be maintained there, and the lower being filled with disulphide of carbon. Outside the cell a massive iron cylinder was placed, having a small central hole pointing towards the line of suspension. The ellipsoid was exposed to the field for several days, and its steady period noted. It was then lowered into the liquid and exposed to the radiation from a strong tube of radium bromide placed in the end of the iron cylinder. After five minutes' exposure the ellipsoid was

Fig. 3.—Testing Cell for exposure to Radium.



raised, allowed to stand in the field for the same time that it had been removed from it, and its period then found to be unchanged. The liquid was used to shield the specimen from any ions which might possibly gather on it from the poles, and to prevent any change in the gradient between the poles by the diffusion of ions from the line of the radium discharge.

If the charge on the ellipsoid had been in any sense free, whether residing on the surface or inside, it would have been rapidly removed and the period lengthened by exposure to the radium. The experiment was repeated with the conditions varied in several ways with always the same result.

It would appear from this that the electrical movement in a dielectric when isolated in the field is entirely confined to the molecule, and is therefore neither metallic nor electrolytic in type, but is the continued displacement of the atomic charges to a greater degree of separation than has been hitherto recognized.

It may be remarked here that the high values of the dielectric constants obtained in the paper are confirmed by measurements on electric cables. Ashton * found that an ozokeritized rubber cable charged at 206·5 volts for 7200 seconds, absorbed $13{\cdot}8 \times 10^{-5}$ coulomb. From the given dimensions the dielectric constant corresponding to this is 13·1. In this case also the slow polarization was elastic in type, as shown by the fact that the quantity discharged on short circuit was just twice that absorbed at constant potential.

It was early suggested by Mossotti that the behaviour of a dielectric can be explained on the assumption that it is made up of isolated conducting spheres. On any of the present electronic theories of matter the separation of charge under the influence of the field gives an arrangement similar to this.

It is now suggested that the first stage of polarization produces such a charged system, but that it differs from a system of conductors in that the charges undergo further separation

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* A. W. Ashton, Phil. Mag. no. 52, 1901, p. 501, "On the Resistance of Dielectrics."

automatically by reason of the attraction and quasi-elastic approach of opposite charges on adjacent molecules. The limit of the first stage of rapid polarization is that of the re-arrangement of atomic charge within the molecule, that of later stages the establishment of electrical strain throughout the mass under new conditions of internal stress set up by the first polarization.

The range of the dielectric constant in the first stage may be taken as the intercept on the vertical axis in fig. 2. Writing this K_0, it is of interest to compare it with the values found for the same substances in alternating fields*, as in Table III. The influence of frequency may from this be anticipated. For those substances in which the ratio is nearly unity it will be at low frequencies small, but for the first few in the list a greater change may be expected between zero and 80 alternations a second than from 80 to the frequency of light.

Table III.

Ratio of the Initial Values of Dielectric Constants in Steady Fields to those in Alternating Fields.

	K_{80}.	K_0.	K_0/K_{80}.
Quartz ‖	4·6	13	2·8
Flint-Glass Δ 4·1	8·5	25	2·9
Resin	3·09	7·0	2·3
Canada Balsam	2·72	19·5	6·9
Paraffin-Wax	2·32	5–6	c. 2·4
Gutta-percha	4·43	7·0	1·58
Sulphur	4·03	4·38	1·07
Indiarubber	3·08	3–4	c. 1·1
Ebonite	2·79	4·2	1·5
Quartz ⊥	4·54	4·55	1·0
Flint Glass Δ 3·3	6·98	7·0	1·0
Amber	2·8	2·8	1·0
Sealing-Wax	4·56	c. 4·6	1·0

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* *Vide* Roy. Soc. Proc. A. vol. lxxxii. p. 422 *et seq.*

ABSTRACT.

The paper contains an account of experiments on the polarization of dielectric ellipsoids and cylinders suspended in a steady electric field. From measurements of the field-intensity, the dimensions of the ellipsoids, and the frequency of torsional swings with and without the field, the dielectric constant can be found from time to time. The method is essentially the same as that using alternating fields described in Roy. Soc. Proc. vol. 82, p. 422. In the present case the longitudinal component of polarization reached a higher value than previously recorded. This was found to be independent of the intensity of the field inside the ellipsoid and to be quasi-elastic in type. The substances examined were quartz, fused and crystalline, flint-glass, amber, sulphur, ebonite, rubber, gutta-percha, paraffin-wax, resin, and sealing-wax. From the rate of increase of the dielectric constant the specific resistance of these was found by considering the change of polarization to be equivalent to a current. The rate of depolarization when the field was reversed was found to be the same as that of polarization and to be uniform for, in most cases, several hours. It is suggested that the cause of this and for the independence of the field-intensity is to be looked for in the continued separation of molecular charge by the attraction of the opposite charges on adjacent molecules induced by the application of the field. From a comparison of the results of the paper and those in alternating fields, the variation of the dielectric constants with frequency can be anticipated.

DISCUSSION.

Mr. JACOB said that some of the Author's curves (fig. 1) showedthat he was dealing with dielectric hysteresis. The results for gutta-percha given in the paper could not be applied in telephonic work because of the high frequencies employed in the latter case. He pointed out that the field strengths used in the paper were much smaller than those occurring in actual work on cables.

Mr. RAYNER doubted whether the method described was accurate to one part in a thousand as stated by the Author. The distance between the electrodes was so large compared with their size, that there would be difficulty in determining accurately the strength of the field at the centre. The difficulties in connexion with the determination of the change of period, especially when it was large, would also be great. He drew attention to the necessity of having the surfaces of the insulators perfectly clean and dry.

Prof. C. H. LEES asked if the variation in the period at the start could be ascribed to the motion of the ions in the vessel onto the dielectric.

Mr. G. L. ADDENBROOKE asked what steps had been taken with regard to the *purity* of the materials employed. He asked also what was the shortest time in which it was possible to get an observation. It would be interesting to know if the Author had investigated the effects described at different temperatures.

Dr. **Russell** stated that he was deeply indebted to Prof. Thornton for his most interesting and instructive paper. The methods of measuring the dielectric coefficient and the insulativity of dielectrics were both novel and valuable. He had a difficulty, however, in understanding why the apparent value of the dielectric coefficient, when the insulating substance was left in an electric field for a long time, did not become infinite. At first sight the experimental results seemed to indicate that it was possible to have a potential difference between points in a dielectric and yet no current. He found it easier to believe that when the steady state was reached there was a minute current flowing across the electrified air to the insulating substance. It was also possible to explain the results by supposing that there was a certain distribution of electric charge inside the dielectric. The subject was one of great practical and theoretical importance, and so he hoped that the Author would continue his researches and clear up the remaining difficulties.

Mr. A. **Campbell** said that the results, which in themselves were highly interesting, were not directly applicable to telephone cables, since the frequency used with these was of the order of 1000 ~ per sec. In connexion with the large polarization shown by gutta-percha, he gave an example of a coil of gutta-percha covered wire which gave an apparent insulation resistance of 200,000 megohms with direct current and 1 minute's electrification, but only a fraction of a megohm when tested at 1000 ~ per second.

Prof. **Thornton**, in reply, said that the very rapid rise of the polarization at the start made it difficult to measure there. The movement of the ellipsoids would have to be recorded photographically. With regard to the accuracy of the method, it was only at low values that it was sensitive. It was found that with electrodes 15 cm. square and 6 cm. apart the field at the centre was uniform. The swings had an average amplitude of 3 degrees. The influence of moisture or surface impurity was recognized at an early stage. The specimens were carefully wiped before suspension, and dried in a warm room *in vacuo* over phosphorus pentoxide for days or weeks before the field was applied. With regard to the influence of ions, it was found that exposing the specimen to radium greatly reduced its apparent polarization. This effect was a double one: the gradient was lowered at the centre of the field and the ions collected around the ends of the ellipsoid, masking the effective charges there. The polarization recorded in the paper always increased, though in the later stage the rate of falling from the linear increase might be in part due to ions. The purity of the quartz and glass specimens was guaranteed by Messrs. Hilger who made them; the paraffin was also pure, the rest were carefully selected.

It is suggested in the paper that the limit to the polarization is that of the separation of charge in the molecule. If the polarization came from a source external to the ellipsoid it always should increase to infinity, and have the same values for different specimens, such for example as quartz cut parallel and perpendicular to the optic axis, for which in fact it is widely different.

XV. *On the Use of Mutual Inductometers.*

By ALBERT CAMPBELL, *B.A.*[*]

(From the National Physical Laboratory.)

§ 1. Introductory.
§ 2. Modified Mutual Inductance Bridge.
§ 3. Measurement of Effective Inductance.
§ 4. Null Method in Iron Testing.
§ 5. Tests of Current Transformers.

§1. *Introductory.*

IN a former paper[†] I have described arrangements, which we may call mutual inductance bridges, by which self inductances, even of very small amount, can be directly measured[‡]. I wish here to discuss further these and other methods in which use is made of variable mutual inductances or, to give them the more convenient name, mutual inductometers.

§ 2. *Modified Mutual Inductance Bridge.*

In the equal arm bridge already discussed there is considerable loss of sensitivity resulting from the insertion of an auxiliary balancing self inductance in one of the arms. By a modification of the arrangement of the bridge, however, the use of the auxiliary coil is rendered unnecessary and the whole apparatus becomes simpler and more efficient. Let us first consider the more general case shown in fig. 1, where the ratio arms are not equal.

Let N be the self inductance of the coil to be measured, r a constant inductance rheostat, L_1 L_2 self inductances, while the current i acts inductively on *both* sides of the bridge as shown, the mutual inductances being M and m. The coils L_1 L_2 may be the upper and lower fixed coils in the inductometer which I have before described (*loc. cit.*), the inducing coils carrying the current i. Thus L_1 and L_2 will usually have mutual inductance (call it y) between themselves.

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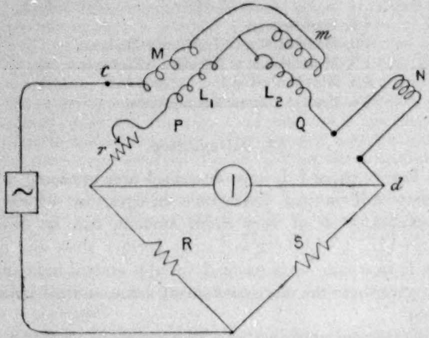
[*] Read January 21, 1910.

[†] Phil. Mag. Jan. 1908, p. 155.

[‡] I would mention here that many years ago Dr. Oliver Heaviside investigated a very general case of inductance bridges (Phil. Mag. p. 173, vol. xxiii. 1887). Most of the possible combinations are included in his paper.

Fig. 1.



Let i_1 and i_2 be the instantaneous values of the currents (of sine wave form) in the arms L_1 and L_2 respectively, the resistances of these arms being P and Q. Let $\alpha = \omega\sqrt{-1}$, where $\omega = 2\pi n$, n being the frequency, and let $R/S = \sigma$.

Then, when there is no current through the galvanometer, we have

$$Ri_1 = Si_2 \quad \text{or} \quad \sigma i_1 = i_2$$

and

$$(P + L_1\alpha)i_1 + M\alpha i + y\alpha i_2$$
$$= (Q + L_2\alpha + N\alpha)i_2 - m\alpha i + y\alpha i_1.$$

Since $i = i_1 + i_2$ we have

$$Pi_1 + L_1\alpha i_1 + M\alpha(i_1 + i_2) + y\alpha i_2$$
$$= Qi_2 + L_2\alpha i_2 + N\alpha i_2 - m\alpha(i_1 + i_2) + y\alpha i_1.$$

Separating the real and imaginary parts, we obtain

$$P = \sigma Q$$

and $$\sigma(L_2 + N) - L_1 = (m + M)(\sigma + 1) - v(\sigma - 1) \quad . \quad . \quad (1)$$

The most useful case of this is when the ratio arms are equal, *i.e.* $\sigma = 1$, and then these equations reduce to

$$P = Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$L_2 - L_1 + N = 2(m + M) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The best arrangement is to make $L_2 = L_1$ permanently in the inductometer, and then the unknown inductance is given directly by

$$N = 2(m + M). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

When L_1 and L_2 are the upper and lower fixed coils in the inductometer, the reading of the instrument is $m + M$ and thus $N = 2 \times$ Reading, and thus is read directly. The sensitivity of the bridge is here much greater than when L_2 consists of a separate balancing coil.

§ 3. *Measurement of Effective Resistance.*

If an alternating current at $n \sim$ per second having an effective value I in flowing through a circuit wastes energy in it at the rate of $R' I^2$ watts (including losses due to eddy currents, magnetic and dielectric hysteresis, &c.), then R' is called the Effective Resistance of the circuit (for frequency n). In general it increases with the frequency, and when telephonic frequencies (500 to 2000 ~ per sec.) are reached, it may often become very much larger than the ohmic resistance. As its value governs the loss of energy, it is of the utmost importance in telephone work to be able to measure it with accuracy. In former papers already cited I have shown how it can be directly measured by a mutual inductance bridge simultaneously with the effective self inductance of the circuit. When the bridge has equal ratio arms which can be interchanged to ensure that they are identical, the method is free from serious error; but if unequal ratio arms have to be employed, large errors may arise from the very small but unavoidable self inductances of these arms. The importance of this in the simple self-inductance bridge was pointed out lately by Giebe *, and upon his mathematical result he based his ingenious method of measuring very small inductances.

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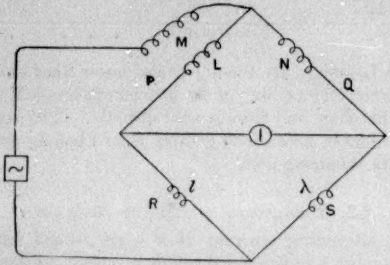
* *Ann. der Physik*, p. 941 (24), 1907.

I have applied similar treatment to the somewhat more complicated case of the mutual inductance bridge as follows.

In fig. 2 let the resistances of the four arms be P, Q, R, and S, and their self inductances L, N, l, and λ respectively.

Fig. 2.



Let M be the mutual inductance as shown. By procedure similar to that in §2 it is easy to show that

$$PS-QR=\omega^2[(L-M)\lambda-(N+M)l] \quad . \quad . \quad (5)$$

and $$SL-RN=(S+R)M-P\lambda+Ql \quad . \quad . \quad . \quad . \quad (6)$$

Several cases are important.

Case (1). When $M=0$ we have

$$PS-QR=\omega^2(L\lambda-Nl) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and $$SL-RN=Ql-P\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

These are Giebe's equations for the ordinary self-inductance bridge.

Case (2). If $R=S$,

then $$(P-Q)R=\omega^2[(L-M)\lambda-(N+M)l]$$

and $$(L-N-2M)R=Ql-P\lambda.$$

If also $\lambda=l$,

then $$(P-Q)R=\omega^2 l(L-N-2M)$$

and $$(Q-P)l=R(L-N-2M).$$

Hence either $R^2 = -\omega^2 l^2$, which is impossible,

or $$P = Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and $$L - N = 2M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Thus we see that in an equal arm bridge the simplest conditions are got by making sure that the ratio arms have equal self inductances. They can be adjusted by interchanging them and altering until the interchange does not alter the balance of the bridge.

Case (3). If R and S are unequal, let $R/S = \sigma$ as before. The arrangement is best used as follows.

Let Q be an auxiliary balancing coil, set to give a balance in the bridge initially when $M = 0$ and $P = P_0$.

Thus we have for the preliminary conditions

$$P_0 S - QR = \omega^2 (L\lambda - Nl) \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

and $$SL - RN = Ql - P_0 \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Now let a coil to be tested, having resistance T and self inductance X, be introduced into the arm P, and let the balance be restored by reducing P_0 to P_1 and by setting M.

Then we have

$$(P_1 + T)S - QR = \omega^2 [(L + X - M)\lambda - (N + M)l]$$

and $$S(L + X) - RN = (S + R)M - (P_1 + T)\lambda + Ql.$$

Subtracting from these (11) and (12) respectively,

$$(P_1 - P_0 + T)S = \omega^2 [(X - M)\lambda - Ml]$$
$$= \omega^2 [X\lambda - M(\lambda + l)]$$

and $$SX = (S + R)M - (P_1 - P_0 + T)\lambda.$$

In the most useful practical case l and λ are small compared with X and M, so we may take as a first approximation

$$SX \doteqdot (S + R)M$$

or $$X \doteqdot (1 + \sigma)M. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Hence

$$P_1 - P_0 + T \doteqdot \omega^2 [\sigma\lambda - l] M/S$$

or $$T = P_0 - P_1 + \omega^2 (\sigma\lambda - l) M/S \quad . \quad . \qquad (14)$$

and a closer approximation then is

$$X \doteq M[1+\sigma-\omega^2(\sigma\lambda-l)\lambda/S] \quad . \quad . \quad . \quad (15)$$

Thus the unknown T and X are obtained from the change made in P, the reading of M, and the corrections due to λ and l. It will be found that the correction is usually almost negligible in the expression for the inductance X. On the contrary, for the effective resistance T the correction may become very important, since ω^2 is large even for moderate frequencies. For example, let $R=99$, $S=1$, $P_0-P_1=20$, $M=1$ millihenry, $l=10$ microhenries, $\lambda=1$ microhenry, and $n=1000 \sim$ per sec., giving $\omega^2 \doteq 40\times10^6$.

Then we have

$$T \doteq P_0-P_1+3{\cdot}6 \doteq 23{\cdot}6 \text{ ohms},$$

while
$$X=100M-3{\cdot}6\times10^{-6}$$
$$=0{\cdot}1-3{\cdot}6\times10^{-6} \text{ henry}.$$

Thus the small inductances l and λ affect the measurement of the effective resistance T by as much as 15 per cent., while the self inductance X is only affected by 3·6 parts in 100,000.

This example shows how much more difficult it usually is to measure effective resistance than self inductance. The difficulty is got over, however, if we can make $R\lambda=Sl$ (*i.e.* $\sigma\lambda=l$), for then the terms involving l and λ disappear and we have

$$X=(1+\sigma)M \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and
$$T=P_0-P_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

To ensure that $R\lambda=Sl$, or in other words that the proportional arms have self inductances in the ratio of their resistances, is not a very easy matter. The following is the best method that I have tried. A coil (A) is constructed of highly stranded wire to give moderately high self inductance (say 0·1 henry) with as high effective insulation resistance as possible *. Its distributed capacity k, which should be as small as possible, may be measured by connecting it through a thermoammeter to a small variable condenser and adjusting the latter to give resonance with an alternating current of known frequency (2000 to 10,000 ∼ per sec.) in a loosely coupled neighbouring coil. If K be the reading of the variable condenser in mfds., and L the inductance in henries of the coil (A), then $n\sqrt{(K+k)L}=159{\cdot}3$, and so k can be found. It is well known that the effective resistance T′ and inductance X′ are given by Dolezalek's formulas,

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* See 'Electrician,' Dec. 10, 1909.

$$X' \doteqdot X(1+2\omega^2 Lk) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$$T' \doteqdot T(1+\omega^2 Lk), \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where X and T are the values if k were absent (in this case the values for very low frequency).

The coil (A) is then tested in the bridge, and if the values obtained for X′ and T′ are not those given by equations (18) and (19), a small amount of inductance is added to R or S until X′ and T′ are read correctly. When this adjustment has been made once for all, it will be found that any other effective resistance will be measured correctly.

It may be mentioned that a well proportioned 0·1 henry coil of 7-strand wire, each strand being of 0·2 mm. diameter and separately insulated, should show practically no variation in effective inductance and resistance due to skin effect at 1000 ∼ per sec.*; its distributed capacity should be less than 0·0001 mfd. If a coil when tested at various frequencies shows percentage variations in its effective resistance which are double those in its inductance, then these variations can be accounted for by distributed capacity; but if the variation of the effective resistance is at a greater relative rate, then we must look for additional causes such as leakage or skin effect (or errors in the bridge).

When the self inductances of the inductometer coils are high, their distributed capacities may cause errors at the higher frequencies. The effects of the capacities of the coils L_1 and L_2 (fig. 1) are got rid of by the method of preliminary balancing; unfortunately this does not eliminate the effect of the capacity k of the external inducing coil. If its resistance and self inductance are ρ and x respectively, then for an equal arm bridge, instead of equations (2) and (4) we have

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* See M. Wien, *Ann. der Physik*, p. 1 (6), 1904, and Dolezalek, *l. c.* p. 1142 (12) 1903.

$$Q=P+2\omega^2k\rho M-\omega^2kM^2 \quad . \quad . \quad . \quad . \quad (20)$$

and

$$N=2M(1+\omega^2kx), \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

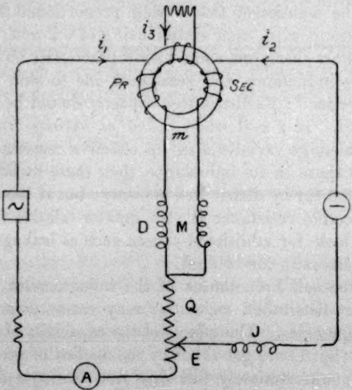
where k is in farads.

The error in Q due to the second term in (20) is usually the most serious. Both errors can be approximately eliminated by connecting a small capacity of suitable amount across the points c, d (fig. 1).

§ 4. *Null Method in Iron Testing.*

The use of a mutual inductometer affords a null method for the magnetic testing of iron; this is analogous to the self inductance method which Max Wien investigated very thoroughly some years ago *, and there are cases in which it may prove of distinct value. The connexions are shown in fig. 3.

Fig. 3.



An iron ring is wound with superimposed primary and secondary coils of turns N_1 and N_2 respectively, m being the mutual inductance between them. These coils are connected, as shown, through the coils of a mutual inductometer D to a source of alternating current and a vibration galvanometer (or tuned telephone), a sliding contact E allowing a part Q to be selected from a resistance connected with the junction of the coils of D. The galvanometer circuit is best made highly inductive by a coil J. Let the hysteresis * and eddy current loss in the ring be represented by a tertiary closed winding, evenly distributed, of resistance R and self inductance L, having mutual inductances F and G to the primary and secondary coils respectively. If $N_1/N_2=b$, then $G=bF$. Let i_1, i_2, i_3 be the instantaneous values of the currents in the primary, secondary, and tertiary coils respectively, I_1, I_2, and I_3 being their effective values. Also let the current i_1 be of sine wave form, which may be attained by electrical tuning or otherwise, the period of the galvanometer being also tuned to that of i_1. The galvanometer deflexion can now be reduced to zero by adjusting D and E to values M and Q respectively.

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* *Ann. der Physik*, p. 859 (66), 1898.

Then we have

$$(R+L\alpha)i_3=F\alpha i_1,$$

and therefore

$$I_3^2(R^2+L^2\omega^2)=F^2\omega^2 I_1^2. \quad . \quad . \quad . \quad (22)$$

Also

$$\begin{aligned}(m-M)\alpha i_1 &= -Qi_1-G\alpha i_3,\\ &= -Qi_1+\frac{FG\omega^2 i_1}{R+L\alpha},\\ &= -Qi_1+\frac{FG\omega^2 R i_1}{R^2+L^2\omega^2}-\frac{FGL\omega^2\alpha i_1}{R^2+L^2\omega^2}.\end{aligned}$$

Hence

$$m-M=-\frac{FGL\omega^2}{R^2+L^2\omega^2},$$

and

$$Q=\frac{FGR\omega^2}{R^2+L^2\omega^2}=\frac{F^2R\omega^2}{b(R^2+L^2\omega^2)}=\frac{RI_3^2}{bI_1^2}.$$

Therefore the total iron loss RI_3^2

$$=QI_1^2 b=QI_1^2 . N_1/N_2. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Also

$$m-M=-QL/R. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

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* If this be not considered rigorous enough for the hysteresis loss, the part relating to it can be proved by another method.

I have assumed that the conditions are not sensibly affected by the small harmonic currents of higher frequency which pass through the galvanometer without any considerable effect on its deflexion. The object of the inductance coil J is to check these, and this could be done even more effectively by adding also a condenser of suitable capacity.

Since the magnetizing current i_1 is of sine wave form, the flux density $\mathcal{B}$ and the total flux Φ will in general not be sinoidal. It is usually necessary to reduce the results of iron tests to a standard value of $\mathcal{B}_{\max}$ with induced secondary voltage of sine wave form. When this is required the $\mathcal{B}_{\max}$ and the form factor of the secondary voltage are observed simultaneously with the above measurement of power. This is done in the usual way by the help of a synchronous commutator [*]. By making tests at two different frequencies the hysteresis and eddy current losses may be separated.

It should be noticed that the work done in the iron by the current i_1 is got by integrating $i_1 d\mathcal{B}$; if i_1 has no harmonics, the harmonics in $\mathcal{B}$ disappear in the integration. I have tried the method with a ring of stampings from ordinary transformer sheet, and obtained a result in very fair agreement with the ordinary wattmeter method; but further experiments are desirable. I have found the method convenient for testing small samples of iron for telephonic work where the tests have to be made for very low values of $\mathcal{H}$ (say about 0·01), and in such cases the method is so sensitive that it is easy to test very small rings weighing only a gram or two. When the eddy currents are very small, equation (24) becomes $m=\mathrm{M}$; and since

$$m=4\pi\times10^{-9}\mathrm{N}_1\mathrm{N}_2\mu s/(\text{circumf. of ring}),$$

where s = section of ring, we can at once find the permeability μ for various values of I_1 (and hence of $\mathcal{H}$). In such cases I have found the values obtained at moderate frequencies in agreement with the results of ballistic tests.

When the eddy currents are large enough to affect the phase of Φ, but do not alter its magnitude appreciably, as Mr. T. L. Eckersley has pointed out to me, we can still find μ; in the galvanometer circuit in fig. 3 the vectors QI_1, ωMI_1, and $\omega N_2\Phi$ form a right-angled triangle (Φ being the effective value), and hence

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[*] See Lloyd, 'Bulletin Bureau of Standards,' p. 467, vol. iv. 1908 also the Author, Proc. Inst. El. Engs. p. 553, vol. xliii. 1909.

$$\omega^2N_2{}^2\Phi^2=(Q^2+\omega^2M^2)I_1{}^2. \quad . \quad . \quad . \quad . \quad (25)$$

Thus we can find Φ and μ for any value of I, by observing Q, M, and the frequency.

§ 5. *Tests of Current Transformers.*

In current transformers used with ammeters, wattmeters, &c., it is necessary to be able to test accurately the ratio of current transformation (I_1/I_2) and the angle of lag (ϕ) between the secondary and the reversed primary current. The above method may be used for this as shown in fig. 4, where the transformer has any desired load; but part of the load is the known resistance S. By considering the galvanometer circuit, when a balance is obtained, the vectors ωMI_1, QI_1, and SI_2 form a right-angled triangle, and hence

$$\tan\phi=\frac{\omega M}{Q} \quad \text{and} \quad I_1{}^2/I_2{}^2=S^2/(Q^2+M^2\omega^2). \quad . \quad . \quad . \quad (26)$$

When ϕ is small, this $\doteqdot\frac{S^2}{Q^2}\left(1-\frac{M^2\omega^2}{Q^2}\right)$,

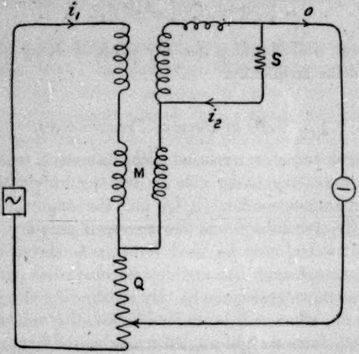
or
$$I_1/I_2\doteqdot\frac{S}{Q}(1-\tfrac{1}{2}\tan^2\phi)$$
$$\doteqdot\frac{S}{Q}\left(1-\frac{\phi^2}{2}\right).$$

Usually ϕ is so small that we may take

$$I_1/I_2=S/Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

With regard to the actual working of the method, the inductometer should be a low-reading one whose primary coils can carry the large primary current. The resistance Q may be a standard low resistance shunted by a slide wire along which the slider runs. Tests by this method on a commercial transformer gave results in practical agreement with those obtained by Mr. C. C. Paterson by a wattmeter method, the source of current being a sine-wave alternator. This agreement is interesting as showing that the harmonics in the secondary current (which are not taken account of by the vibration galvanometer) have very slight effect on the results. The method should only be used where the primary current can be made approximately sinoidal.

Fig. 4.



In conclusion, I would express my best thanks to Dr. Glazebrook for his kind interest in the work, and to Mr. T. L. Eckersley for valued and suggestive criticism.

ABSTRACT.

In the use of mutual inductometers (or variable inductances) already described by the Author, the use of a balancing coil in one arm of the bridge causes considerable loss of sensitivity. With an equal-arm bridge this difficulty is overcome by putting the two halves of the secondary circuit in adjacent arms of the bridge. The auxiliary balancing coil is thus dispensed with and the usual formula is still applicable.

The Author next discusses the measurement of effective resistance, which is in general much more troublesome than that of self-inductance. As the effective resistance determines the total power spent by a given

alternating current in a conductor, it is a most important quantity in telephonic and other high-frequency work. When it is measured by an ordinary self-inductance bridge, Giebe has shown that large errors may be introduced by the small residual inductances of the ratio arms. The Author works out the analogous formulas for mutual inductance bridges, which indicate that the inductances of the ratio arms must be accurately proportional to their resistances, if errors are to be avoided.

He next describes a null method in iron testing analogous to Max Wien's self-inductance method. The ring to be tested is wound with primary and secondary coils. The magnetizing current I_1 is passed through the primary coil, the primary circuit of a mutual inductometer, and a slide-wire resistance. The detecting instrument, a vibration galvanometer or a tuned telephone, is put across a circuit consisting of the secondaries of the ring and the inductometer in opposition and a part Q of the slide-wire resistance. By adjusting Q and the reading M of the inductometer a balance is obtained, in which case the power lost in the ring (due to hysteresis and eddy currents) is equal to $QI_1^2 \times N_1/N_2$, where N_1 and N_2 are the numbers of turns in the windings of the ring. In certain cases the permeability can also be directly found. The method is immediately applicable to the testing of current transformers. If the instrument usually in the secondary circuit of the transformer be replaced by a suitable low resistance S, then when a balance is obtained

$$\tan\phi = 2\pi n M/Q \quad \text{and} \quad I_1/I_2 \doteqdot S/Q,$$

where ϕ is the angle of lag between the primary and the reversed secondary current, and n is the frequency. The primary current should have a sine wave form. The method gives directly the two quantities wanted in practice, but, owing to considerations of wave form, the results must be interpreted with caution.

DISCUSSION.

Mr. C. C. **Paterson** congratulated the Author upon his null method of iron testing and pointed out some of its advantages.

Mr. W. **Duddell** remarked with regard to the self-inductance of the ratio arms that the time constants of the two should be the same, and suggested the use of woven resistances.

Mr. **Campbell**, in reply to Mr. Duddell, stated that when the ratio arms of the bridge could be of high resistance the employment of woven resistances could ensure that their time constants were equal. The main difficulty, however, arose with ratio arms of relatively low resistance (*e.g.* 10 : 1000 ohms); for such values woven resistances did not appear quite applicable.

XVI. *Electrical Recording Thermometers for Clinical Work. By* H. L. Callendar, *M.A., LL.D., F.R.S., Professor of Physics at the Imperial College of Science and Technology, S.W.*[*]

In Memoriam Dr. A. GAMGEE, F.R.S., *obiit* Mar. 29, 1909.

1. The scientific interest and importance of a method of continuously recording the temperature of the body, whether in health or disease, is now being generally recognized, and I have been persuaded that my experiments in the construction and testing of suitable apparatus for the purpose may be of some value at the present juncture, although I have no results of physiological importance to produce.

My attention was first directed to the subject by Professor J. G. Adami, F.R.S., of McGill College, Montreal, at whose suggestion I constructed some electrical resistance thermometers specially adapted for recording the temperature of different parts of the suface of the body. Unfortunately I had to leave Montreal early in 1898, before the apparatus had taken its final shape, and the results obtained at that time, consisting merely of experimental records of my own normal temperature, did not seem to be of sufficient interest to merit publication. I did not return to the subject until November 1908, when I undertook to make some thermometers for Dr. A. Gamgee, F.R.S., and to collaborate with him in evolving a practical method suitable for general use in clinical work. His paper, then recently published, "*On Methods for the Continuous* (*Photographic*) *and Quasi-Continuous Registration of the Diurnal Curve of the Temperature of the Animal Body*"[†], dealt exclusively with thermoelectric methods of temperature measurement: but he had become convinced that the thermoelectric method was unsuitable for general use, on account of the delicacy of the apparatus, and the necessity for employing an elaborate thermostat. The Copper-Constantan thermocouples which he employed gave an E.M.F. of 40 microvolts per 1° C., which was ample for delicate photographic methods of registration, but insufficient for the more robust type of instrument required for general use with ink records. A more serious difficulty in practice was the necessity of keeping one of the junctions of the couple at a constant temperature in the neighbourhood of 37° C. for long periods. In the application of the resistance method no thermostat was required, and it was easy to obtain a tenfold larger deflexion with the same galvanometer.

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[*] Presidential Address, February 11, 1910.

[†] Phil. Trans. Roy. Soc. B. vol. 200, p. 219 (1908).

The objections that have commonly been urged against electrical resistance thermometers for this kind of work are (1) that resistance thermometers are more difficult to construct and to insulate satisfactorily than thermocouples, and (2) that their indications are liable to be disturbed by the heating effect of the current employed. These objections undoubtedly exist, and have frequently proved fatal to the employment of resistance thermometers : but they have arisen chiefly from faulty application or construction, and not from defects inherent in the method. Objection (1) is readily surmounted by proper methods of construction, and objection (2) by a proper consideration of the conditions of sensitiveness.

Conditions of Sensitiveness.

2. The conditions of sensitiveness in measuring a resistance by the Wheatstone bridge method have been discussed by Maxwell ('Electricity and Magnetism,' vol. i. p. 437) and Heaviside (Phil. Mag. Feb. 1873), whose results have generally been quoted and applied to the problem under consideration. They start with the assumption that the battery power available is limited by the internal resistance of the cells, and give rules for obtaining the maximum current through the galvanometer with this implied limitation. The limitation of battery power was no doubt an important consideration in many kinds of telegraph testing thirty years ago, but it is rarely applicable in modern laboratory practice, and never in dealing with electrical resistance thermometers. In the majority of resistance measurements, and more particularly in electrical thermometry, the limiting condition is imposed by the heating effect of the current on the resistance to be measured, and the resistances, etc., should be chosen and arranged to give the greatest

sensitiveness for a given limiting value of the current through the resistance to be measured. The resistance of the battery circuit is quite immaterial provided that the battery can be arranged to give the required limiting current. The problem has been discussed from this point of view by Guye (*Arch. Sci. Phys. Nat. Geneva*, 1892), and by Schuster (Phil. Mag. xxxix. p. 175, 1893), and in several of my own papers, but it will facilitate discussion to reproduce here the investigation itself in a simplified form.

Let the annexed diagram, fig. 1, represent the arrangement of resistances in a Wheatstone bridge, in which C is the

Fig. 1.

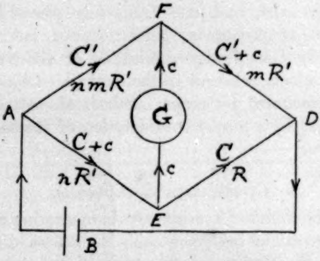

Diagram of Wheatstone Bridge.

current through the resistance to be measured R, and c is the current through the galvanometer of resistance G, when the bridge is not balanced. The resistance in series with R is nR' traversed by a current $C+c$; the resistance in parallel with R on the same side of the galvanometer circuit is mR' traversed by a current $C'+c$. The resistance in the opposite arm of the bridge to R is nmR', traversed by a current C'. The currents are assumed to flow in the directions indicated by the arrows, and it is evident that they satisfy the condition of continuity. The numbers n and m, representing the ratios of the arms of the bridge, may have any positive values. The bridge is balanced when $R=R'$, in which case $c=0$, and $C'=C/m$.

Since the difference of potential between the ends of the galvanometer circuit is Gc, we have by Ohm's law,

$$RC - mR'(C' + c) = Gc = nmR'C' - nR'(C + c).$$

Eliminating C', we obtain for the ratio c/C,

$$c/C = (R - R')/(G(1 + n)/n + (1 + m)R'). \quad . \quad . \quad (1)$$

It is obvious that this ratio, which may be regarded as a measure of the sensitiveness of the arrangement, is quite independent of the resistance or E.M.F. in the battery circuit. It is also at once evident that, for a given defect of balance as measured by $R - R'$, the value of the ratio c/C will be a maximum when n is as large as possible, and m as small as possible, the limiting value of the ratio c/C in this case being, when

$$n = \infty, \text{ and } m = 0, \; c/C = (R - R')/(G + R').$$

This point has been emphasized by both Guye and Schuster, and comes out clearly in Heaviside's investigation.

The advantage which can be gained by making n large and m small is not very great, because when $n = m = 1$,

$$c/C = (R - R')/2(G + R'),$$

or the sensitiveness is still half as great as in the limiting case.

For practical purposes it is more important to observe that if n is small or m large, the sensitiveness may be indefinitely reduced. This arrangement should always be avoided if possible, as shown by the following example.

Maxwell's Rule.

3. Maxwell has given a rule which is very often quoted and applied ('Electricity and Magnetism,' vol. i. section 348). "Of the two resistances, that of the battery and that of the galvanometer, connect the greater resistance so as to join the two greatest to the two least of the four other resistances."

This rule is seldom applicable to accurate resistance measurements, and when applied in platinum thermometry has often led to disastrous results. A simple numerical example will make this clearer.

Suppose that it is required to measure the resistance of a platinum thermometer, $R=10$ ohms, with a post-office box having ratio arms 10 and 1000 ohms. Suppose that the galvanometer has a resistance $G=10$ ohms, and that the battery is a 2 volt cell of negligible resistance. According to Maxwell's rule we should connect the galvanometer so as to join the two resistances of 1000 ohms to the two resistances of 10 ohms, in which case we should have $n=1$, and $m=100$ in fig. 1. Suppose that the balance is slightly disturbed by a change $dR=R-R'$. Substituting in equation (1) we obtain:

Case (A), Maxwell's Rule,

$n=1$, $m=100$, $c/C=dR/1030$, $C=0{\cdot}1$ amp.

If, however, we interchange the connexions of the battery and galvanometer, which is simply equivalent to interchanging the values of n and m in the equation, we obtain for the Anti-Maxwell arrangement:

Case (B), Anti-Maxwell,

$n=100$, $m=1$, $c/C=dR/30$, $C={\cdot}002$ amp.

The actual value of the current through the galvanometer in case (A) is greater than in case (B) in the proportion of 3 to 2. Maxwell's rule is so far justified. But the current through the resistance to be measured is 50 times greater in (A) than in (B), and the heating effect of the current is 2500 times greater. With an ordinary platinum thermometer arrangement (A) might give an elevation of temperature of about 5° C., which would be fatal for any purpose of measurement. Whereas arrangement (B), with nearly the same sensitiveness, would give an elevation of ·002° C. only, which is sufficiently small for the most accurate work.

The essential point is to observe that the ratio c/C of the galvanometer current to that through the thermometer, which is the true measure of sensitiveness in this case, is 34 times greater in (B) than in (A). (B) is therefore by far the better arrangement in the case where the heating effect of the current on the resistance to be measured is the primary consideration.

The rule to replace Maxwell's in this case, if there is any choice as to the arrangement of the battery and galvanometer connexions, is: "Connect the battery so as to make the resistance in series with the thermometer greater than the resistance in parallel." This will make n greater than m in fig. 1, and it is evident that the ratio c/C will be diminished or the sensitiveness reduced, if n and m are interchanged.

In the practical use of platinum thermometers, we have the further restriction that n must be equal to unity, because it is necessary to compensate the changes of resistance of the leads by the equal changes of resistance of a pair of compensating leads on the opposite side of the galvanometer contact on the bridge-wire. The loss of sensitiveness, as compared with the case of n very large, does not amount to more than 20 or 30 per cent., and is of no consequence as compared with the trouble and uncertainty involved in measuring the resistance of the leads separately at each observation. Moreover, if n is made large, the heating effect of the current C on the resistance nR' may become serious. As a matter of fact this is a common and very insidious source of error in the use of a P.O. box with $n=100$, especially if the coils are of german silver or platinoid. With maganin resistances the effect is much less marked.

It is still possible to gain some advantage in point of sensitiveness by making m smaller than unity, provided that it is not made so small that the heating effect of the current C/m on the resistance mR' becomes appreciable. For the most accurate work I have generally employed thermometers having a resistance about 25·6 ohms at 0° C., with a fundamental interval of 10 ohms, and have made the ratio coils about 6·4 ohms, giving $m=1/4$ *, which gives an advantage

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* The reason for choosing this particular value of m in the case of the compensated box with platinum-silver and platinum coils, which was exhibited at the Royal Society in May, 1893, and fully described later (Phil. Trans. R. S. A., vol. 199, p. 92), was that the change of resistance of the platinum-silver coils due to the heating effect of the current might be of the same order of magnitude as that of the platinum coils, and might take effect at the same rate. The thermal capacities and radiating surfaces of the coils being nearly equal, but the temperature coefficient of the platinum-silver nearly 16 times less than that of the platinum, the platinum-silver coils would carry about 4 times as great a current as the platinum for the same change of resistance. This was found to give a fairly safe limit for the value of m, but the effect was so small as to be of little importance.

of about 20 per cent. in point of sensitiveness as compared with $m=1$. A more important consideration in this case is that the same box can be used with thermometers of lower resistance, *e.g.* $R_0=2\cdot56$ ohms, F.I.$=1$ ohm, without serious loss of sensitiveness.

Resistance of Galvanometer.

4. It is generally possible to vary the resistance of a galvanometer through a wide range, by altering the connexions of the coils or by substituting one coil for another, without altering the mass of the coils. In this case the sensitiveness will vary approximately as the square root of the resistance G. We see from equation (1) that the deflexion of the galvanometer, which is proportional to $c\sqrt{G}$, will be a maximum when

$$G=n(1+m)R'/(1+n).$$

This result agrees with that given by Maxwell, and shows that in the majority of cases which occur in practice, G should be of the same order of magnitude as R, being restricted to the limits 2R and R/2, if n is never less than 1, or m greater than 1. If we are restricted, as in platinum thermometry, to the case $n=1$, we should take

$$G=(1+m)R'/2.$$

The best resistance for the galvanometer is restricted to the same limits, 2R and R/2, provided that m does not exceed 3.

For example, in the case of the box already quoted, with ratio coils 6·4 ohms, when used with a thermometer $R_0=25\cdot6$ ohms, the best resistance for the galvanometer was 16 ohms. When the same box was used with a thermometer ten times smaller, $R_0=2\cdot56$ ohms, the best resistance for the galvanometer would be 4·5 ohms. This could be secured approximately by putting the two coils of the 16 ohm galvanometer in parallel, but the advantage gained thereby would be only 10 per cent. If, on the other hand, the ratio coils had been made equal to 25·6 ohms each, according to the rule given by Maxwell and generally followed, there would have been a loss of sensitiveness of 20 per cent. with the 25·6 ohm thermometers, and 50 per cent. with the 2·56 ohm thermometers, which could not so well be neglected.

Resistance of the Thermometer.

5. For a platinum thermometer of resistance R_0 at 0° C., the change of resistance per 1° C. is approximately $\cdot 004\ R_0$, which must be substituted in equation (1) for the value of dR or $R-R'$, in order to find the deflexion of the galvanometer per degree change of temperature. Making this substitution, and remembering that $n=1$, we find that, when the resistance of the thermometer is changed, the deflexion per degree, which is proportional to $c\sqrt{G}$, varies as $RC\sqrt{G}/(2G+(1+m)R)$.

The rise of temperature produced in the thermometer by the measuring current C is directly proportional to C^2R, and inversely proportional to the rate of dissipation of heat per degree rise of temperature of the thermometer above its surroundings. The rate of dissipation of heat depends on the form and surface of the thermometer, and on the conditions of exposure. For similar thermometers of different resistances, but of the same size, under similar conditions of exposure, we must have C^2R the same in order to secure the same degree of accuracy as limited by the heating effect of the current. The permissible current C will therefore vary inversely as the square root of the resistance of the thermometer. Making this substitution, we have

$$c\sqrt{G} \text{ varies as } \sqrt{GR}/(2G+(1+m)R).$$

We see from this result that, if it is possible to choose the best resistance for the galvanometer, namely $(1+m)R/2$, and if m is the same for all, the sensitiveness and accuracy of the thermometer, so far as the heating effect of the current is concerned, will be independent of its resistance. In accurate laboratory work, where a delicate galvanometer is available, the heating effect of the current is so small as to be relatively unimportant, and the resistance of the thermometer is chosen chiefly with a view to minimise errors due to defects of insulation or imperfect contacts, or small differences in the resistance of the thermometer and compensator leads. On the other hand, for practical applications, or for ink recording, it is necessary to employ a fairly robust and portable galvanometer of the suspended or pivotted coil type; the current employed must be considerably increased if a large scale record is required, and it may be importan to choose the resistance of the thermometer to suit the galvanometer or *vice versa*, in order to minimise the heating effect of the current.

The symmetry of the above expression in respect of G and R shows that the condition governing the choice of the resistance of the thermometer, if the galvanometer is given and m is constant, is the same as that already given for the galvanometer in terms of R, namely,

$$R=2G/(1+m).$$

If, however, the value of $mR=S$ is constant, being fixed by the resistance of the ratio coils, each equal to S, of the bridge employed, the maximum sensitiveness is obtained when

$$R=2G+S.$$

The latter condition gives the limit beyond which no advantage can be gained by increasing R and diminishing m, even if the heating effect of the current on S can be entirely neglected. If the heating effect of the current in the thermometer is kept constant, as already assumed, the heating effect in S increases in proportion to R/S. On the same assumption the total current to be supplied by the battery (which is a consideration in records of long duration) is a minimum when $R=S$, but is only increased in the ratio 5/4 when $R=4S$. The maximum sensitiveness is obtained when $S=0$ and $R=2G$, but the sensitiveness will be reduced by little more than 10 per cent. if we make $S=G/2$, keeping $R=2G$. It would be reduced a further 20 per cent. if we made $R=G=S$, according to Maxwell's investigation *.

In employing the deflexion method described below, the advantage of making R larger and S smaller than G is really greater than would appear at first sight, because, is the sensitiveness is reduced 20 per cent. for the same heating effect of the current, it means in practice that the current must be increased 20 per cent. to give the required deflexion, or that the heating effect of the current must be increased more than 40 per cent. The most important error to avoid is to make R small compared with G or S. The sensitiveness may in this way be very greatly reduced, and the heating effect of the current increased five or ten times. Thus, a pyrometer of low resistance designed for work at high temperatures (where the resistance is high and an open scale is not required) would not be a suitable instrument to employ for recording small variations of temperature on an open scale at low temperatures, unless G and S were made inconveniently small, and the pyrometer itself specially wound with extra thick wire to give a large cooling surface.

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* Maxwell, 'Electricity and Magnetism,' vol. i. § 349. This is the right arrangement if all the four arms of the bridge are similar, and equally affected by the current.

Construction of Thermometers.

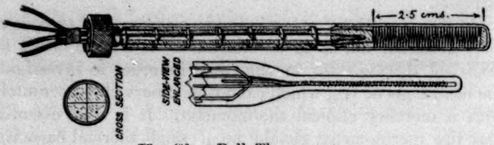
6. The construction of the thermometers depends on the situation in which they are to be used. I have made three principal types:—(1) For insertion in the mouth; (2) for the rectum; (3) for the surface of the skin or for the axilla (armpit).

(1) The mouth is not a suitable position for records of long duration, but it is of interest to be able to insert an electrical thermometer in this way in order to investigate the conditions of lag, which cannot be observed so accurately with a mercury clinical thermometer. It is most essential that the thermometer should be of small thermal capacity, in order to minimise the effect of inserting a cold thermometer, since the tissues of the body are not very good conductors of heat. In any case, if the mouth has not been kept shut for some time previous to the insertion of the thermometer, there will be a large apparent lag due to the recovery of temperature of the mouth itself. A so-called half-minute clinical thermometer may take upwards of five minutes to get within a half a degree of the true temperature under these conditions. I have tested several recording thermometers from this point of view, and the results are of some interest.

Flat Glass Bulb Thermometer.—The ordinary type of platinum thermometer has the wire wound on a mica cross with compensated leads insulated by mica disks. Thermometer and leads are slipped into a containing tube about 10 mm. diam., which is removable, and can easily be replaced if damaged or broken, and which permits withdrawal of the thermometer for adjustment or repair, if required. This type is most convenient and suitable for ordinary laboratory work with sensitive galvanometers; but is insufficiently sensitive for insertion in the mouth. Moreover, since the wire is surrounded by air, the heating effect of the current may be excessive when such a thermometer is used with recording instruments on an open scale, especially if its resistance is low.

A great improvement on this type, in point of quickness of action and diminution of the heating effect of the current, is readily effected by winding the wire on a *flat* plate of mica in place of a cross, and melting the lower part of the containing tube (which is preferably of lead glass) down on to the wire so as to form a flat bulb, as illustrated in fig. 2. If the glass is thin, the sensitiveness may be increased nearly five times, as compared with a thermometer of the ordinary type, and the heating effect of the current reduced in nearly the same proportion. When inserted in a water-bath at 37° C. this thermometer takes less than a minute in arriving within a hundredth of a degree of the final temperature; but when inserted cold in the mouth, it may take 4 or 5 minutes to get within a tenth of a degree, because the tissues of the mouth take time to recover from the cooling effect of inserting the thermometer, as further illustrated below. Where quickness of action is an essential condition, as in some kinds of calorimetric work, the flat-bulb thermometer is a great improvement on the ordinary type with a round tube, but it shares with the mercury thermometer the disadvantage that, if broken or damaged, it cannot easily be repaired.

Fig. 2.



Flat Glass Bulb Thermometer.

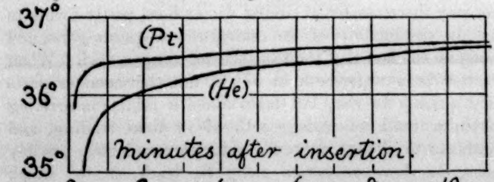
It is essential that thermometers of this type should be provided with compensated leads, otherwise there will be a variable immersion error, and an apparent lag due to slow conduction of heat along the leads. It is also most important for accurate work to avoid screw terminals in the head of the thermometer, and to provide each instrument with flexible leads two or three metres long, permanently soldered to the thermometer and compensator leads, and securely attached to the head of the thermometer, as indicated in fig. 2.

Heræus Quartz-Glass Thermometer.—The smallest size of quartz-glass resistance thermometer constructed by Heræus after the experiments of Dr. Haagn has a bulb 25 mm. long and 3 mm. in diameter, and is therefore very similar in dimensions to an ordinary clinical thermometer. The bulb contains a coil of fine platinum wire wound on a rod of quartz-glass, and protected by a very thin tube of quartz-glass fused over it. The bulb is fused on to a larger quartz tube, containing the two leading wires, which are of gold. The resistance of the coil is 25 ohms nearly at 0° C., but, as there are no compensating leads, the resistance cannot be measured very accurately. This type of thermometer is very suitable for insertion in the mouth, but is unsuitable for records of long duration owing to its shape and the risk of breakage. It does not appear to be quite so sensitive as a mercury thermometer of similar dimensions, partly owing to the low conductivity of the central rod of quartz-glass, and partly to the fact that the leads are not compensated. When inserted in a water-bath at 37° C. this thermometer starts very quickly to rise, but takes about a minute in arriving within a tenth of a degree C. of its final reading, and continues to change appreciably for three minutes, possibly owing to slow conduction along the leads, one of which passes through the centre of the coil. When placed under the tongue, after keeping the mouth closed for ten minutes previously, it arrives within a degree of the final temperature in one minute, but takes seven or eight minutes to get within a tenth of a degree. A typical record obtained with this thermometer is given in fig. 3.

Flat Platinum Tube Thermometer.—This is a type of thermometer which I specially designed for calorimetric work, where quickness of action and small mass are important. It is similar to the flat glass bulb thermometer, except that the containing tube is of platinum about 0·05 mm. thick, which permits the thermal capacity of the bulb to be greatly reduced. The coil is wound on a flat plate of very thin mica, and insulated from the flattened platinum tube by thin strips of mica on either side. The lag of this thermometer when placed in a water-bath at 37° C. was so much less than that of the galvanometer that it could not be recorded satisfactorily. Owing to its size and shape, it was not very suitable for insertion in the mouth, as it could not be placed in the usual position under the tongue. When placed along the side of the mouth (after closure for ten minutes) between the tongue and the teeth, it arrived within 0°·3 of the final temperature in less than a minute, but continued to rise appreciably for nearly ten minutes, apparently indicating that the mouth had not been kept closed long enough beforehand to reach a practically steady temperature.

A comparison of the records of the platinum tube thermometer (Pt) and the Heræus quartz tube thermometer (He) when placed in the mouth under similar conditions is shown

Fig. 3.



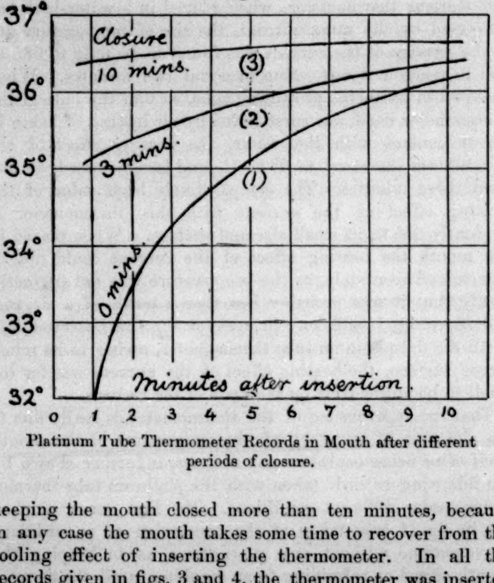
Comparison of Quartz and Platinum Tube Thermometers in Mouth after closure for 10 minutes.

in fig. 3, which illustrates the importance of extreme quickness and small thermal capacity in such tests.

It might appear at first sight as though the slow and long continued rise indicated by both thermometers in different degrees, were due to the heating effect of the measuring current, which might easily produce a result of this nature if the current were excessive or the mass considerable. But the current employed in taking these records scarcely exceeded one hundredth of an ampere in the case of either thermometer, and the heating effect of the current was measured in both cases and found to be very small. With the Heræus thermometer, when placed in a water-bath and traversed by the same current, the rise of temperature due to the passage of the current was found to be only 0°·085 C. and to reach a steady value in about two minutes. When enclosed in its metal protecting tube, so that the bulb of the thermometer itself was surrounded by air instead of being in direct contact with the water, the heating effect of the current was increased to 0°·31 C., and became steady in less than three minutes. The comparatively high value of the heating effect of the current with this thermometer is evidently due to its small size and surface. When placed in the mouth the heating effect of the current could not be determined accurately, as the temperature was not perfectly steady, but it was probably less than a tenth of a degree, and therefore insufficient to account for the observed lag. With the flat platinum tube thermometer, owing to its much larger surface, the heating effect of the current was far too small to be appreciable on the scale of the records.

That the apparent lag of the thermometer is really due to the slowness of recovery of the temperature of the mouth itself after being cooled in any manner, is further shown by the following records taken with the platinum tube thermometer under different conditions. The first curve, marked (1) in fig. 4 was taken at the conclusion of an ordinary conversation, without any precautions as to keeping the mouth closed or refraining from speaking, until the thermometer had been inserted. The thermometer did not reach 32° C. for more than a minute, and was still nearly 1° C. below the normal temperature after a lapse of seven minutes. Curve No. (2) represents the record obtained after keeping the mouth closed for three minutes previous to the insertion of the thermometer. Curve No. (3), which is practically a reproduction of the record already given in fig. 3, represents the result of keeping the mouth closed for ten minutes previously. There is a further improvement with this thermometer after closure for fifteen minutes, but unless the mass of the thermometer is very small, or the thermometer itself is previously warmed, little advantage is gained by

Fig. 4.



Platinum Tube Thermometer Records in Mouth after different periods of closure.

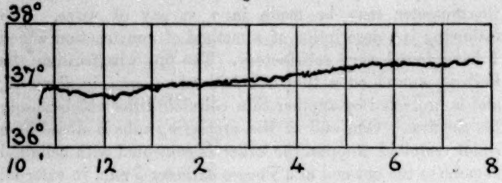
keeping the mouth closed more than ten minutes, because in any case the mouth takes some time to recover from the cooling effect of inserting the thermometer. In all the records given in figs. 3 and 4, the thermometer was inserted cold, being taken direct from a water-bath at 15° C. The difference between the records in fig. 4 is due almost entirely to an actual difference in the temperature of the mouth at starting. The difference between the quartz and platinum tube thermometers in fig. 3 is due chiefly to the greater cooling effect of the quartz tube, owing to its greater mass in proportion to its surface.

(2) For records of long duration, the best results are undoubtedly to be obtained by insertion in the rectum. The thermometer bulb should be of small dimensions and the leading wires should be enclosed in a flexible but inextensible tube of small diameter, such that it can be attached to the bulb in a perfectly secure and water-tight manner. Such a thermometer may be made in a variety of ways. The following is a description of a method of construction which I have found very satisfactory. The fine wire forming the bulb is wound on a thin celluloid tube 5 mm. in diameter, and is protected by another thin celluloid tube which closely fits the first. One end of the protecting tube is closed by a small celluloid stopper, the other is cemented with celluloid cement to the cut end of a Porges catheter 5 mm. in external diameter, and 32 cm. long, through which the flexible thermometer and compensator leads are passed. This type of catheter has a smooth surface, and is very flexible, but practically inextensible, thus protecting the leading wires and joints from possible strain. It also makes a very good joint with the celluloid tube protecting the bulb. I have tried rubber tubes and metal bulbs, but these appear to be in every respect less suitable. A rubber stopper fixed on the catheter at a distance of 3 to 5 cm. from the near end of the bulb, is a convenient method of limiting the depth of insertion. The thermometer is smeared with vaseline before insertion. This prevents irritation, and does not appear to have any injurious effect on the celluloid or catheter.

The curves shown in figs. 5 and 6 are typical records taken with this thermometer on consecutive nights under similar conditions. In the first case there was an unusually rapid rise in the external atmospheric temperature during the night which occasioned some feeling of discomfort. In the second case there was a sharp frost, and the patient awoke feeling appreciably chilled. From a comparison of similar records it would appear that the normal temperature of the body may be influenced to some extent, as one would naturally expect, by the external conditions prevailing at the time. There is often a fall of nearly half a degree C. during the night corresponding more or less with the diurnal change of atmospheric temperature.

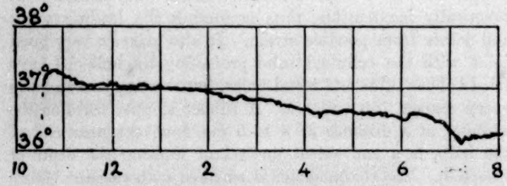
The changes here observed cannot have been due, as might appear at first sight, to the effect of change of temperature on the resistance coils of the box or measuring apparatus, because increase of resistance of the balancing coils in the bridge would act in the opposite direction, producing an apparent fall of the thermometer. Moreover, the balancing coil in this particular thermometer was made of fine manganin wire having a resistance equal to that of the thermometer coil at 37° C., and was placed inside the celluloid tube on which the thermometer coil itself was wound, so that any changes in its resistance were due to change of temperature of the thermometer, and were allowed for when the thermometer was calibrated. This construction was adopted in order to avoid the necessity of applying any correction for change of temperature of the resistance box; but it is open to the objection that it makes the thermometer rather less sensitive, and doubles the heating effect of the current. The

Fig. 5.



Rectal Thermometer on Thread Recorder.

Fig. 6.



Rectal Thermometer on Thread Recorder.

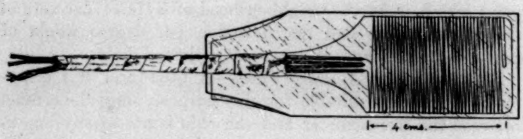
sensitiveness of the celluloid rectal thermometer, though only about half as great as that of the Heræus thermometer, was ample for records of long duration. The heating effect of the current was only a tenth of a degree, and did not introduce any appreciable error, since it remained practically constant, and was allowed for in the calibration, in which the same current was employed.

A more important source of systematic error would lie in variation of E.M F. of the battery during a long continued run. But unless the storage cells employed were nearly run down, this would not be likely to amount to more than 1 per cent. of the galvanometer deflexion, or two or three hundredths of a degree, since the galvanometer deflexion corresponds only to the small difference from the balance point which is in the neighbourhood of 37° C. The zero of the galvanometer and the deflexion per degree would of course be tested, and, if necessary, adjusted daily.

(3) *Surface or Axillary Type.*—A tube form of thermometer is not very suitable for this purpose, since the contact between the thermometer and the skin is necessarily imperfect. It is desirable to give the thermometer as large a surface as possible, in order to diminish the heating effect of the current, and at the same time to make the bulb flexible and thin, so as to conform approximately to the contour of the surface to which it is applied. The first thermometer of this type which I constructed for Prof. Adami of Montreal in 1897, was made by sticking a thin bolometer grid of platinum foil on to a thin sheet of celluloid photographic film. This satisfied the required conditions very perfectly, but the bolometer grid was excessively fragile (unless its resistance were made unduly low), and was also difficult to attach securely to the leading wires. The substitution of fine wire for the foil greatly facilitated the construction and adjustment, and was found to make little, if any, difference to the sensitiveness. In any case it is most important that the thermometer should be fairly robust, as it has to undergo somewhat rough usage. In my experience nothing answers so well as celluloid film for the insulating material. The wire may be protected by cementing over it a thin film of celluloid; but a better plan is to insert the thermometer in

a thin flat celluloid sheath, as illustrated in fig. 7, which can be changed at any time if it becomes soiled. The flexible leads are flattened and insulated between celluloid films at the end where they join the thermometer coil. In applying the thermometer to the surface of the body, a truss, similar to that employed by Dr. Gamgee (*loc. cit.*) for thermocouples, may be used ; but, in my experience, a simpler and more generally applicable method is to back the thermometer with a pad of cotton-wool to secure good contact, and to keep it in place by means of an elastic band to which the thermometer and leads are pinned or sewn. When the thermometer is placed in the left axilla, which is generally the most convenient location, the elastic band should pass under the left arm, and over the right shoulder near the neck.

Fig. 7.

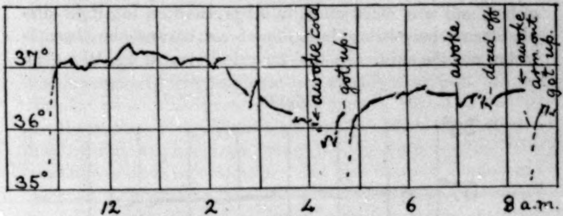


Axilla or Surface Thermometer (Celluloid).

A surface thermometer of this type is extremely sensitive, and shows no appreciable lag when placed in a water-bath. But when placed in the axilla, or elsewhere on the surface of the skin, it may take fifteen minutes or more to reach a steady temperature. The reason of this is partly that the skin is slightly chilled by exposure during the insertion of the thermometer, and partly that the application of the thermometer with its wool pad tends to raise the temperature of the skin locally by preventing evaporation. The temperature of the skin under normal conditions with free evaporation is generally lower than that of the body. If the patient is already in bed and well covered up, the initial lag may be greatly reduced, but is in any case of little or no consequence for records of long duration.

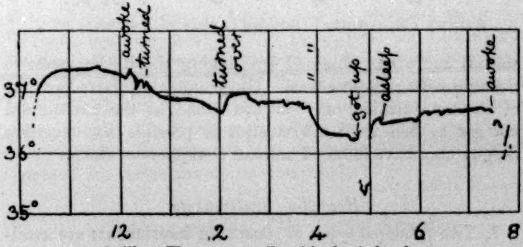
A thermometer of this type should not be heated to 100° C. or directly exposed to water at temperatures as high as 50° C., because this procedure is liable to buckle the film. If the temperature coefficient of the wire is known, it can be calibrated with sufficient accuracy by a single observation in a water-bath at the ordinary room temperature with a reliable mercury thermometer. If it is to be tested in a water-bath at 40° or 50° C., it should be protected by a thin copper sheath. I have generally employed thermometers adjusted

Fig. 8.



Axillary Thermometer A, with elastic band.

Fig. 9.

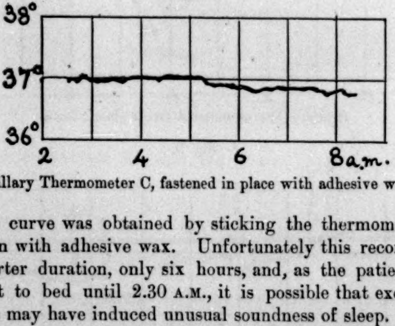


Axillary Thermometer B, with elastic band.

to a fundamental interval of 10 ohms, giving a change of one-tenth of an ohm per 1° C. Each thermometer is provided with a balancing coil, the resistance of which is adjusted to be equal to that of the thermometer at or near 37° C.

The chief difficulty in obtaining reliable records of the body temperature by means of a thermometer inserted in the axilla, is that the indications are liable to be disturbed by movements of the patient, more particularly by putting the left arm out of bed, or by raising the bed-clothes and admitting cold air in turning over from one side to the other. These effects are illustrated by the records shown in figs. 8 and 9, which were taken with different thermometers in the axilla on consecutive nights. The curve is often very smooth during sleep, but becomes irregular owing to greater restlessness on awaking. These accidental excursions seldom exceed a few tenths of a degree, unless the patient gets out of bed, and are chiefly due, in all probability, to slight displacements permitting intrusion of air betwen the thermometer and the skin. In the record shown in fig. 10 a very smooth curve was obtained by sticking the thermometer to the skin with adhesive wax. Unfortunately this record was of shorter duration, only six hours, and, as the patient did not get to bed until 2.30 A.M., it is possible that excessive fatigue may have induced unusual soundness of sleep.

Fig. 10.



Axillary Thermometer C, fastened in place with adhesive wax.

Recording Instruments.

7. Two principal types of recording instruments are available, corresponding respectively to the Deflexion Method, and the Balance Method of measuring variations of electrical resistance. In the deflexion method, the greater part of the resistance to be measured is balanced on a Wheatstone bridge by a balancing coil, and the small variations are observed by the deflexion of the galvanometer. In the balance method,

the galvanometer contact is moved along a bridge-wire until balance is obtained. This method is not so quick for small variations, but is generally most suitable for work with resistance thermometers, and has the advantage that the scale is uniform and independent of the E.M.F. of the battery. In clinical thermometry, the range of temperature to be covered is so small that the deflexion method is generally applicable and the scale practically uniform. The scale is readily adjusted to read in degrees of temperature by means of a suitable rheostat in the battery circuit. If storage cells of sufficient capacity are employed there is little risk of error from variation of the E.M.F. of the battery. Thus, for clinical thermometry, there is little to choose between the balance and deflexion methods, if the instruments employed in either case are equally sensitive. In the installation of underground thermometers, buried at various depths in the soil at McGill College in 1894, the deflexion method was employed, as being the quickest, for the daily readings of each thermometer ; but the balance method was adopted, as being the most accurate, for taking continuous records of the diurnal variations when occasion required. In order to obtain a strictly continuous record by the deflexion method it is necessary to employ photography, but a practically continuous record may be obtained, as in the Thread Recorder made by the Instrument Co., Cambridge, by depressing the pointer attached to the galvanometer coil so as to mark its position on the record sheet by means of an inked thread at regular intervals of a minute, the index being free for the greater part of the time. For records of long duration the separate dots coalesce into a continuous curve if the variations are not too rapid.

Adjustment and Use of the Thread Recorder.

8. The standard type of Thread Recorder is a fairly robust instrument, having a resistance of about 10–12 ohms, and a scale of 80 mm. to one millivolt. With a Copper-Constantan thermocouple, this instrument gives a scale of about 3 mm. to 1° C., which is rather too small for clinical records. With a resistance thermometer, it is easy to get a scale of 50 mm.

to 1° C. if desired, but a scale of 10 to 20 mm. is usually sufficient.

In using the thread recorder with a resistance thermometer, some form of platinum thermometer bridge with a slide-wire is required, and a rheostat or resistance box capable of fine adjustment between 30 and 50 ohms. This rheostat is connected in series with a 2 volt storage cell to the battery terminals (marked B on the bridge), and serves to adjust the scale of the record. The thread recorder is connected to the terminals marked G on the bridge, one of which is connected between the equal ratio coils of 10 ohms each, and the other to the sliding contact on the bridge-wire, as shown on the diagram, fig. 11.

Fig. 11.

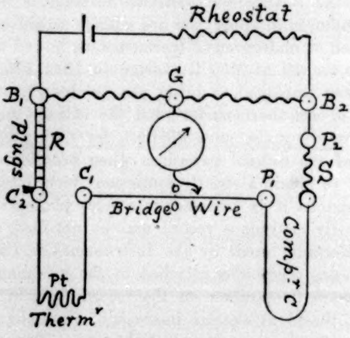


Diagram of Connexions for Thread Recorder.

The two thermometer leads are *not* connected to the terminals marked P on the bridge (which is the usual arrangement when the balance method is used), but to the terminals marked C_1 C_2 (for compensator) so as to be in series with the adjustable plug resistances denoted by R in the annexed diagram of connexions. The compensator leads are connected in series with the Balancing Coil, denoted by S in the diagram, to the terminals marked P_1 P_2. With this arrangement the resistance S + C is nearly constant, so that if the galvanometer deflexion is adjusted to be correct when the thermometer is at 0° C. or 20° C., or any convenient temperature, the scale will be very nearly correct when the thermometer is at any other temperature, the plugs R being suitably adjusted.

As an example of the adjustment of the galvanometer scale, we will suppose that the thread recorder is properly set up and levelled, and the zero adjusted to the right hand side of the scale, which reads from right to left. The standard type of record sheet is divided into 25 scale-divisions, with a heavy line at each fifth division. With this sheet it is most convenient to adjust the scale to read 5 divisions to 1° C., giving a range of 5° C. on the record sheet. Suppose that the thermometer has a resistance of 26 ohms at 0° C., and is adjusted to have a fundamental interval of exactly 10 ohms, giving a change of 0·1 ohm per 1° C. Suppose that the balancing resistance S is 30 ohms and is adjusted to be equal to the thermometer at 40° C. Ten centimetres of the bridge-wire, or ten units of the plug resistances R, will correspond to a tenth of an ohm, or 1° C., on the thermometer.

To adjust the galvanometer scale, the thermometer may be placed in a vessel of water at the temperature of the room, and the bridge balanced. If the temperature is 15° C. (which is 25° C. below 40° C.) it will be necessary to unplug 250 units (equivalent to 25°) in the box at R to balance S, when the sliding contact is at zero in the centre of the bridge-wire. If balance were obtained with 230 units in R, and the sliding contact at 7·3 cm. to the right of zero, it would mean that the temperature of the water was $40 - 23 + \cdot 73 = 17^{\circ}\cdot 73$ C. In any case, to obtain a fine adjustment for the balance, the sliding contact must be shifted till the galvanometer shows no deflexion, and then clamped in position. The galvanometer being thus balanced at a steady temperature, increase the resistance unplugged in R by 50 units (0·5 ohm), corresponding to a rise of temperature of 5° C. on the thermometer. If the adjustment of the rheostat is correct the galvanometer should deflect 5° to the left, *i. e.* to the extreme left of the record sheet. If the galvanometer deflects to the right the connexions of its terminals must be reversed. If the deflexion is less than 5°, the resistance of the sliding rheostat must be reduced, and *vice versa*, until the desired deflexion is obtained. The adjustment will then be correct for any temperature within the range of the instrument up to 40° C. To get a range on the record sheet from 35° to 40° C., for records of normal body temperature, it would then be necessary to unplug 50 units in R and to set the bridge-wire contact at zero, so that the zero of the galvanometer scale may correspond to 35° C. Similarly for a range from 40° to 45° C., the resistance unplugged in R should be zero. Temperatures above 45° could be measured by increasing the balancing resistance S, but it is not advisable to go above 45° C. with celluloid thermometers.

The accuracy of adjustment of the thermometer and its balancing coil S can be tested at any time by immersing the thermometer either in melting ice, or in water at the room temperature if a reliable mercury thermometer is available. If, for instance, the galvanometer reads zero when the thermometer is in water at 17°·67 C. with 230 units unplugged, and the contact at 73 mm. (reading 17°·73 C.), the thermometer reads 6 mm. or ·06° C. too high at this temperature, and the error may be corrected by setting the bridge-wire contact 6 mm. to the right of zero for the record, instead of at zero. Several different thermometers may be used with the same balancing coil, provided that allowance is made in this way for the small differences between them.

Testing for Defects.

9. To test the thermometer and compensator leads for defective insulation disconnect one of the thermometer leads, say C_2, from the battery, and the opposite compensator lead, say P_1, from the bridge-wire (or *vice versa*, C_1 from the bridge-wire and P_2 from the battery). If, now, the galvanometer circuit is made, there should be no deflexion if the insulation is perfect, since the bridge-wire and galvanometer contact are effectively isolated from the rest of the circuit, unless there is defective insulation between the thermometer and compensator. Before disconnecting the wires for this test the galvanometer circuit should be broken, otherwise the

instrument will be violently agitated, and its zero may be affected.

To measure the heating effect of the working current on the thermometer, place the thermometer in water at a steady temperature, balance the bridge as already explained, and adjust the galvanometer scale if this has not been done. With the galvanometer balanced at zero, and the current adjusted to its proper value, change the battery from one storage cell to two cells of the same E.M.F. Since the resistance of a storage cell is negligible compared with the other resistances in the circuit, this change will have the effect of doubling the current through the thermometer. The rise of temperature of the thermometer due to the passage of the current will be quadrupled, since it varies as the square of the current. The change of resistance of the thermometer due to changing from one cell to two, will be three times the change of resistance due to the heating effect of the current with one cell. The deflexion of the galvanometer for a given change of resistance will also be doubled, since the current is doubled. The deflexion of the galvanometer produced by changing from one cell to two will therefore be six times the heating effect with one cell measured in degrees of temperature. Hence observe the galvanometer deflexion in degrees of temperature and divide by six to find the heating effect for the normal current with one cell. If the heating effect so deduced does not exceed a tenth of a degree, it will not introduce any appreciable error, since it remains practically constant, so long as the scale adjustment is the same. With a given thermometer and galvanometer the heating effect varies as the square of the number of scale divisions per degree. Thus if the heating effect were found to be a tenth of a degree with the galvanometer adjusted for a scale of 2 cm. to 1° C., it would be four tenths if the scale were increased to 4 cm. per degree, and the zero would be raised three tenths.

Calibration of the Scale of the Galvanometer.

The angular deflexion of the galvanometer for one millivolt is rather large, and the scale may not be accurately one of equal parts. This should be tested for accurate work.

With the aid of the bridge it is easy to make the test, either by placing the thermometer in water at a steady temperature, or by substituting a fixed resistance, equal to the balancing coil, in place of the thermometer. Balance the galvanometer at zero by adjusting the bridge-wire contact. Adjust the scale to the required value. Unplug successively resistances in R corresponding to 1, 2, 3, 4, 5°, and observe the galvanometer reading for each on the record sheet. Repeat in descending order to test the galvanometer for change of zero due to imperfect elasticity of the suspension. The errors should not as a rule be appreciable on the scale of the record unless there is something wrong with the levelling or suspension. It is of course extremely important that the galvanometer coil should be perfectly free in all positions of the boom, and it must be remembered that a variation of level may cause a variation of scale, as well as of zero.

Balance Method with Slide-wire Recorder.

10. This is the type of recorder generally employed with resistance thermometers, especially where a large range of temperature is to be covered. It is equally applicable for clinical work, or for small ranges of temperature, provided that a suitable thermometer is employed, and that the current is properly adjusted to avoid excessive heating of the thermometer. It gives a strictly continuous ink record, with a pen attached to the sliding contact on the bridge-wire, which is automatically maintained at the balance point by means of a pair of motor clocks actuated by the deflexion of the galvanometer. The galvanometer coil carries a light arm terminating in a contact fork consisting of two short platinum wires close together. A wheel with a platinum edge rotates between these two wires, one or other of which makes contact with the wheel (according to the direction of deflexion of the galvanometer), and completes a relay circuit actuating one or other of the motor clocks. This form of relay is incredibly delicate, and has since been adapted by Brown for the purpose of submarine telegraphy. The contacts keep themselves clean automatically, and never require adjustment. Since the deflexion of the galvanometer is limited to a fraction of a millimetre, there is no

chance of accidental disturbance of the zero by excessive deflexion.

The scale of the instrument is independent of the E.M.F. of the battery, being determined by the resistance of the bridge-wire in relation to that of the thermometer. The bridge-wires are usually made with a scale of 2·0, 1·0, or 0·5 ohm to a length of 20 cm., which is the length of the record sheet. With a thermometer having a zero resistance of 26 ohms, and a fundamental interval of 10 ohms, these bridge-wires give ranges of 20° C., 10° C., and 5° C., respectively on the record sheet, or scales of 1, 2, and 4 cm. to 1° C. With thermometers of smaller resistances the scales are proportionately smaller and the ranges larger. For clinical work the most convenient range is from 35° to 45° C., with a scale of 2 cm. to 1° C. This is obtained with a 26 ohm thermometer and a 1 ohm bridge-wire.

A recorder of this type is more expensive than the thread recorder, but is complete in itself, with the exception of the battery, and does not require either an auxiliary resistance box, or a rheostat for adjusting the scale. It is a great advantage in practice that the scale never requires adjustment, but is always correct to about 1 in 1000, provided that the bridge-wire is correct and uniform. It is easy to change from one scale to another when required by changing the bridge-wire, which gives a range of 1 to 4, or by changing the thermometer, which gives a range of 1 to 20, from a fundamental interval of 10 ohms to a fundamental interval of 0·5 ohm.

For ordinary work thermometers are generally provided with an "ice-bobbin" or balancing coil, equal in resistance to the thermometer at 0° C. The ice-bobbin for each thermometer is connected to its appropriate terminals when the thermometer to which it belongs is in use. If the thermometer is required to cover an extensive range of temperature, a series of auxiliary resistances, generally ranging from 0 to 20 ohms is provided, which enables the range to be extended to 20 times the range of the 1 ohm bridge-wire. With a 26 ohm thermometer the range thus obtained would be 200° C., or 2000° C. with a 2·6 ohm pyrometer. For clinical work these auxiliary resistances may be dispensed with if the balancing resistance is adjusted to 30 ohms to balance the thermometer at 40° C., and the zero fixed at the centre of the bridge-wire so as to give a range of from 35° to 45° C., with a 1 ohm bridge-wire.

Some observers have experienced trouble with this type of recorder owing to the heating effect of the measuring current. Such troubles have arisen chiefly from the employment of unsuitable thermometers with excessive currents. With a 26 ohm thermometer the instrument can be adjusted to work perfectly on a scale of 4 cm. to 1° C. with a current of one hundredth of an ampere through the thermometer. The record can be read easily to less than a hundredth of a degree, or 1 in 30,000 ; and the heating effect of the current, with a celluloid thermometer of the surface type above described, is only two or three thousandths of a degree, or practically inappreciable on the most open scale of the record.

Testing and Adjustment of Slide-wire Recorder.

11. The most important points to test in a slide-wire recorder are the adjustment of the zero of the galvanometer, and the zero of the slide-wire. When the galvanometer coil is free and the instrument levelled, the two prongs of the contact fork should be just clear of the contact wheel on either side. If there is any torsion in the suspension, one of the prongs will bear against the wheel, and the pen will travel continuously in one direction when the circuit of the relay magnets is made, the galvanometer circuit being open. The torsion head should be adjusted until the pen does not travel either way under these conditions. This is a very delicate test. If there is much torsion in the suspension, the apparent zero on the slide-wire will vary to some extent with the resistance in circuit when the galvanometer circuit is closed. This adjustment is made before the instrument is sent out, but it is desirable to test it occasionally, especially if the suspension has been strained or damaged, or if a new suspension is fitted.

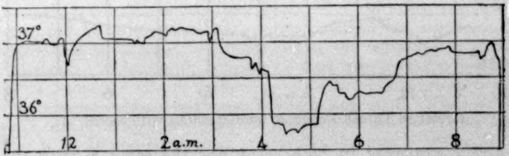
To adjust the zero of the slide-wire, short circuit the pyrometer and compensator terminals, PP and CC, with short equal pieces of copper wire. When the battery is switched on the pen will quickly come to rest at the zero.

If the position is not quite correct in reference to the record sheet, the slide-wire may be shifted in the direction of its length through a distance equal to the observed deviation. For many purposes it is most convenient to have the zero in the centre of the slide-wire. The wire can then be changed, if desired, for another of different resistance and scale, without altering the zero. If it is desired to have the zero at one end of the scale, it is necessary to provide a small auxiliary slide for correcting the adjustment when the slide-wire is changed. The freedom of the galvanometer suspension from torsion may conveniently be tested at the same time as the zero of the slide-wire, by doubling the resistance in the battery circuit, so as to halve the current through the slide-wire. If this does not produce any appreciable shift of zero, the suspension must be very nearly free from torsion. It is important that the external resistance added should be only in the bridge circuit, and not in that of the relay magnets, which will not work satisfactorily if the P.D. on the relay circuit is much reduced. This mistake has often been made in using this type of recorder.

Slide-wire Records.

12. It is possible to obtain records on a more open scale with the slide-wire recorder than with the thread recorder, owing to the greater width of the record sheet, which is 20 cm. for the slide-wire recorder, as compared with 8 cm. for the thread recorder. The latter is limited by the range of the galvanometer boom, which cannot be indefinitely elongated without making the deflexion too sluggish. The record reproduced in fig. 12 was taken with an axillary thermo-

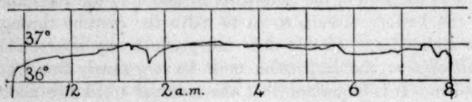
Fig. 12.



Axillary Thermometer D, on Slide-wire Recorder, 4 cm./deg.

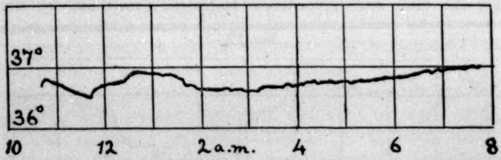
meter differing in a few details of construction only from those employed for records 8 and 9. The scale of the original record was 4 cm. per 1° C., which is 2·5 times the scale adopted with the thread recorder. The scale has been reduced to one third, to facilitate reproduction and comparison. It will be observed that the record shows the same characteristic dip, culminating between 4 and 5 A.M., as records 8 and 9. When first observed, it seemed as though this dip of temperature might be due to some instrumental error, but it has been observed on several occasions with this particular patient, with different thermometers and different recording instruments. The dip is not always present. The record reproduced in fig. 13, which shows an extremely steady temperature, was taken on the same patient, with the same recorder and thermometer, but on a scale of 2 cm. per 1° C., similarly reduced in reproduction. The current employed in both cases was the same as with the thread recorder for records 8 and 9, and the thermometer was attached in precisely the same way. The only effect of reducing the scale with the slide-wire recorder would be to make the instrument proportionately quicker in responding, which cannot account for the greater smoothness of the

Fig. 13.



Axillary Thermometer D, on Slide-wire Recorder, 2 cm./deg.

Fig. 14.



Rectal on Thread Recorder (simultaneous with record 13).

record. A record taken simultaneously on the same patient with the rectal thermometer and thread recorder, is reproduced in fig. 14. The variations shown are very slight, and the differences between the two thermometers are such as would be likely to exist in different parts of the body.

The records above given are sufficient to show, if proof were needed, that accurate clinical records of temperature on a fairly open scale under practical conditions may readily be obtained by the aid of electrical resistance thermometers, with either of the recording instruments above described. The recording instruments are standard types, which have been thoroughly tested by the experience of a great variety of users for many years, and are not likely to give trouble.

ABSTRACT.

The PRESIDENT dealt with the application of resistance thermometers to the recording of clinical temperatures. Records of clinical temperatures have been obtained by Dr. Gamgee using a thermocouple in conjunction with sensitive recording instruments. The objections to thermocouples are twofold. In the first place, the E.M.F.'s developed are so small that the recording instruments must be very sensitive and therefore unsuitable for ordinary use. In the second place, serious difficulties arise with regard to the thermostat which is necessary to maintain one of the junctions of the thermocouple at a constant temperature. The chief difficulty in connexion with the use of resistance thermometers for this kind of work lies in the heating effect of the current. In order to explain how this difficulty is overcome Prof. Callendar gave a short account of the conditions upon which the sensitiveness of a Wheatstone's bridge depends. He pointed out that in platinum thermometry, in order to obtain accurate compensation for the resistance of the leads, it is necessary that the ratio arms of the bridge should be equal, and he showed that this condition reduced the sensitiveness, which could be obtained by suitably varying the resistances by about 30 per cent. In joining up a bridge in work with resistance thermometers, Maxwell's rule for the positions of the battery and galvanometer which give maximum sensitiveness is seldom applicable. While Maxwell's arrangement actually gives the greatest sensitiveness, the heating effect of the current is so much greater that the trouble arising from this cause more than counterbalances the increased sensitiveness. The problem to be solved in designing a suitable thermometer for clinical work is, with a given galvanometer and resistance-box, to find the resistance of the thermometer which will give the most accurate results for a given heating effect of the current. This is given by the equation $R=2G+S$, where G is the resistance of the galvanometer and S that of one of the

ratio arms. In the apparatus used at the meeting G=S=10 ohms giving R=30 ohms, a resistance for the thermometer which gives the convenient scale of 0·1 ohm increase per degree Centigrade. It is important in the construction of a thermometer for clinical work to secure quickness of action and to reduce the heating effect of the current. An ordinary tube-form of thermometer is good for laboratory work with sensitive galvanometers, but it is unsuitable for use with recorders. An ordinary form of thermometer takes about 1½ minutes to reach a steady temperature when taken from water at the temperature of the room and placed in water at 0° C., whereas a thermometer suitably designed will take up a steady temperature in about ten seconds. The pattern of the thermometer must also be suited to the purpose for which it is intended. Three types were shown, designed for mouth, rectal, and surface work. Experiments were performed proving that these thermometers fulfilled the necessary conditions. The recording instruments for use with these thermometers were briefly described. Prof. Callendar then showed continuous records obtained from a patient with a normal temperature. The temperature is generally very steady if the thermometer does not shift or the patient get wholly or partly out of bed. The effects of external changes of temperature were also shown, and simultaneous records taken on different parts of the body illustrated the fact that the temperature does not vary in the same way at all places. On the medical side there is a vast field for research which may eventually result in important additions to knowledge.

XVII. *Telephone Circuits.*

By Professor JOHN PERRY, *F.R.S.*[*]

ABOUT a year ago Mr. Sidney Brown told me that he wished to place contrivances at equal distances in telegraphic or telephonic circuits and that it was necessary to make calculations, beforehand, because in a telephone line, say, of three hundred miles or more with similar contrivances inserted every one or two miles, it was practically impossible to proceed by any kind of experimental adjustment. He spoke of various kinds of contrivance involving combinations of condensers and inductance coils and transformers and even rotary motors and generators; some in series with the line and others as shunts to earth.

Exact investigation of what occurs in telephonic and telegraphic signalling in general is quite impossible. Trans-

[*] Read February 25, 1910.

mitting and receiving instruments are various in their complexity ; signals give rise to all sorts of functions of time and space ; cables are not uniform in character, and when they are loaded with inductances, capacities, &c., the expressions employed in exact mathematical analysis are too complex for even the greatest mathematicians to deal with.

Ordinary people like myself ought to keep on the lower levels and not talk about " waves " in such problems. The fundamental equation for v or c in an ordinary cable is exactly the same as that for the conduction of heat through a solid bounded by parallel plane faces ; in the heat problem nobody dreams of talking about waves. It is easy to convert the differential equations into wave equations, and in the hands of exceptionally clever men the physical ideas involved become important. Even when Mr. Brown's problem is taken up in a mere mathematical way it is troublesome, and I can offer no exact solutions. I have, however, found a simple method of making the kind of calculation required, correct enough for all practical purposes and the method can be applied by non-mathematical persons. I mean that the necessary calculations may be made by any person who understands formulæ sufficiently well to be able to apply arithmetic to them. The idea came to me at St. Louis when I was listening to a paper by Prof. Kennelly, but I then had no interest in pursuing the subject. Prof. Kennelly has published many illustrations of the great value of the use of *cosh* and *sinh* functions ; he gets no more credit now than Columbus did after he showed how to make the egg stand upright.

Let me say that much of the arithmetic consists in converting such an expression as $\alpha + \beta i$, where i means $\sqrt{-1}$, to the shape $r(\cos\theta + i\sin\theta)$, which is conveniently written $r(\theta)$. I need hardly say that if this is raised to the nth power the result is $r^n(n\theta)$ where n may be negative or fractional. Also when $r(\theta)$ operates upon such a function of the time as $a\sin qt$ it converts it into $ar\sin(qt+\theta)$. $\operatorname{Cosh} r(\theta)$ and $\sinh r(\theta)$ will be found tabulated (for $\theta = 45°$) by Prof. Kennelly. Unfortunately for our purposes we use only small values of r for which Prof. Kennelly's tables have only a few entries. But calculation is easy enough.

Write $r(\theta)$ in the shape $\alpha+\beta i$. Then

$$\cosh r(\theta) = \cosh\alpha \,.\, \cos\beta + i\sinh\alpha \,.\, \sin\beta$$

and

$$\sinh r(\theta) = \sinh\alpha \,.\, \cos\beta + i\cosh\alpha \,.\, \sin\beta,$$

so that the calculation is easy for any values of α and β.

I will first make a few statements, the foundation of my reasoning.

1. It is well known that any looped line or cable containing a receiving instrument at the end of the loop has its equivalent in a cable whose far end goes to earth through the receiving instrument, the outer coating of the cable being everywhere to earth or of zero potential. The mathematical proof of this is very easy and I shall not give it, although I do not know that it has been published.

2. In all difficult problems cables are so long that for all practical purposes we may suppose them to be infinitely long. Thus it is well known to practical people that when lines of 30, 40, &c. miles of the same cable are experimented upon, the rate of diminution of current per mile is the same except for regions near the ends, and these end effects are much the same for all lengths of line except in short lines.

3. In any long uniform cable at equidistant points A, B, C, &c., when a simple periodic current is flowing, its value at these points being C_A, C_B, C_C, &c., the ratio of C_A to C_B is the same as that of C_B to C_C, &c. This is true except near the ends.

4. In a long cable with perfectly similar contrivances, A_1A_2, B_1B_2, C_1C_2, &c., at equidistant places A, B, C, &c.; when a simple periodic current is flowing, its value at corresponding points being A_1, A_2; B_1, B_2, ; C_1, C_2, &c., the ratio of A_1 to B_1 is the same as that of B_1 to C_1, &c., and these are also the same as the ratio of A_2 to B_2, B_2 to C_2, &c. We may say the same about the voltages at corresponding points. It is seen that I assume an infinite line, but even in the case of an infinite line my assumption is that there is *some* diminution in the amplitude of the current from A_1 to B_1. If the circuit behaved like a mere Ohmic resistance the current would be the same everywhere but the voltage would not.

5. Neither telephone nor telegraph signalling can be

studied correctly through the behaviour of periodic currents, but it has been found in practice that in telephony, if we assume that all currents are periodic functions of the time, the frequencies being not much less than 600 nor much greater than 1000 per second, the most important being 800, good transmission of such currents is found to mean good transmission of speech.

6. It is quite a usual thing to find that experimenters are using transmitting and receiving devices which are not the best for a particular cable. For a receiving instrument, for example, there is a best resistance and inductance (or combination of inductance, capacity, back E.M.F., &c.). In every case yet studied by me (but I can imagine others) the most suitable receiver causes the current and voltage near that end of the line to be different from what they would be if the line were infinitely long. Thus, for example, with simply periodic currents in the standard cable it will be found that at any place far from the ends, voltage divided by current is of the form $r(-45^c)$ or $\alpha - \alpha i$. If the telephone or other receiving instrument has a resistance R and an inductance L each of which is proportional to the square of the number of turns of wire upon it, and if the effect on the instrument (such as the deflexion of a coil, &c.) is proportional to the current turns received, this effect is a maximum if $\mathrm{R} = \alpha$ and if $\mathrm{L} = \alpha/q$, where q is 2π times the frequency. It is easy to show that if we wished the receiving instrument to produce the same effect as an infinite prolongation of the cable, it ought to have a resistance R and a negative inductance $\mathrm{L} = -\alpha/q$, which is a capacity $1/\alpha q$. It is then not merely when unsuitable instruments are used, but also when the most suitable instruments are used that we have end effects.

7. A line with equidistant similar contrivances when they are not too far apart may be replaced (mathematically) by a line whose properties (as to resistance, inductance, capacity, leakance, &c.) are continuous, for many purposes of calculation.

The simplest case of uniform distribution of properties in an infinite line was studied in my paper read before the Physical Society in 1893. Let v and c be the voltage and current at any place whose distance from the sending end is x; let $v_0 \sin qt$ be the voltage when x is 0. If r, l, s, and k are the resistance, inductance, leakance, and capacity per unit length, we have

$$v = v_0 \epsilon^{-hx} \sin (qt - gx), \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\sqrt{\frac{kqr}{2}} \sqrt{\sqrt{\left(1+\frac{q^2l^2}{r^2}\right)\left(1+\frac{s^2}{k^2q^2}\right)} \mp \left(\frac{ql}{r} - \frac{s}{qk}\right)} \quad (2)$$

is the value of h if the minus sign be taken and the value of g if the plus sign be taken. The following formula for the current c is useful—

$$c = \sqrt{\frac{h^2+g^2}{r^2+q^2l^2}}\, v_0 \epsilon^{-hx} \sin \left(qt - gx + \tan^{-1}\frac{g}{h} - \tan^{-1}\frac{lq}{r}\right). \quad (3)$$

I showed in tables how good signalling was affected by changes in the values of the constants and specially by alterations in l and s. There was nothing very new in my paper, but it enabled the non-mathematical man to understand some of the discoveries of Mr. Heaviside. At that time I saw no easy way of calculating how the introduction of detached equidistant contrivances caused discrepance from the idea of continuously distributed properties.

In much that follows, lq will be so large compared with r that, taking $s=0$, we have

$$h = \frac{r}{2}\sqrt{\frac{k}{l}}, \quad g = q\sqrt{kl}.$$

If s is small but not negligible, g is as before, but

$$h = \frac{1}{2}\left(r + \frac{sl}{k}\right)\sqrt{\frac{k}{l}}.$$

As for other uniform distribution of properties, it is only necessary here to speak of those which may immediately be put in the above shape. For example, besides having merely r and l in series in the line, we may have something of the nature (mathematically) of k_1 farads per mile, so that instead of $r + lqi$ we have

$$r + lqi + \frac{1}{k_1 q_1} \quad \text{or} \quad r + qi\left(l - \frac{1}{k_1 q^2}\right).$$

This is as if we had merely the resistance r and an inductance

$$l' = l - \frac{1}{k_1 q^2}.$$

Again, instead of the leakance s which is the reciprocal of a resistance R, let us have also something of the nature of an inductance L, say, and assume R to be small relatively to $\mathrm{L}q$. Then instead of s we have

$$\frac{1}{\mathrm{R} + \mathrm{L}qi} = \frac{\mathrm{R}}{\mathrm{L}^2 q^2} - \frac{i}{\mathrm{L}q}.$$

Thus the common expression $s + kqi$ becomes

$$\frac{\mathrm{R}}{\mathrm{L}^2 q^2} + qi\left(k - \frac{1}{\mathrm{L}q^2}\right).$$

For preliminary calculations it is generally sufficiently correct to assume $\mathrm{R}/\mathrm{L}^2 q^2$ to be zero.

We may now consider that the telephone line has, per mile, a resistance r, an inductance $l' = l - \frac{1}{k_1 q^2}$, a leakance $s = 0$, and a capacity $k' = k - \frac{1}{\mathrm{L}q^2}$. It will be found desirable in almost all cases to have l' so large that we may take

$$h = \frac{r}{2}\sqrt{\frac{k'}{l'}} \quad \text{and} \quad g = q\sqrt{k'l'}.$$

Assuming these conditions to exist we have, very nearly,

$$c = \sqrt{\frac{k'}{l'}}\, v_0 \epsilon^{-\frac{1}{2} r x \sqrt{k'/l'}} \sin\left(qt - qx\sqrt{k'l'}\right).$$

For a line of, say, L miles in length the amplitude of the arriving current is a maximum if $\sqrt{\frac{l'}{k'}} = \frac{1}{2} r\mathrm{L}$. In fact the arriving current is the sending current divided by ϵ. It will be found that in practice we never have so small a value of h as this gives, and we generally try to make h as small as possible.

We can approximate to the above conditions by inserting contrivances at equal distances in the line. I will take a few examples. I prefer just now to consider a cable in which I have recently taken an interest; $r = 18$ ohms per mile,

$k = \cdot 055 \times 10^{-6}$ farad per mile, $s=0$, $l=0$. I take q to be 5000 (that is, a frequency of 800).

1. This cable with no contrivances,

$$h = g = \sqrt{kqr/2} = \cdot 04975.$$

So that if x is 1 mile, $\epsilon^{-hx} = 0\cdot 9513$. That is, there is an attenuation of 5 per cent per mile.

2. Let $l' = l = 0$. That is, there is neither inductance nor capacity inserted in the line. Now let there be an inductance leak to earth which causes k' to be $0\cdot 01375 \times 10^{-6}$. Then

$$h = q = \sqrt{k'qr/2} = \cdot 024875,$$

so that $\epsilon^{-h} = \cdot 97513$, or there is an attenuation of 2·55 per cent. per mile. The approximation to this is by an inductance to earth of L henries per mile, where

$$\cdot 055 \times 10^{-6} - \frac{1}{\mathrm{L}q^2} = \cdot 01375 \times 10^{-6},$$

or $\mathrm{L} = \cdot 972$ henry. When I use an inductance to earth I generally prefer to put it in series with a condenser. Thus, instead of an inductance of 0·972 henry, I would use an inductance of 1·012 henries in series with a condenser of 1 microfarad. This enables the line to be tested more easily.

Instead of an inductance leak of 0·972 henry per mile we may insert $0\cdot 972/m$ henry every m miles. As m gets greater the discrepance from the continuous conditions gets greater. The problem to be solved, later, is in all cases, to find the discrepance for a particular value of m and what therefore is the most convenient distance by which contrivances ought to be separated.

3. Suppose $l' = -\cdot 09112$ and $k' = -\cdot 0125 \times 10^{-6}$. These values will cause c to be a maximum if the distance is 300 miles. It will be found that they mean, a capacity of $0\cdot 439 \times 10^{-6}$ farad per mile in series with the line and an inductance leak of 0·5925 henry per mile. Here $g = 0\cdot 16775$ and $h = \frac{1}{2}\sqrt{k'/l'} = 0\cdot 00333$. This means an attenuation of 0·333 per cent. per mile when q is 5000.

Now when q is taken as 3000 or 4000 the attenuation is 0·64 or 0·53 per cent. per mile.

The approximation to these conditions is (*a*) a contrivance every m miles consisting of a capacity of $0{\cdot}439/m$ microfarad in series with the line and an inductance leak of $0{\cdot}5925/m$ henry; or (*b*) a contrivance every m miles consisting of a capacity of $0{\cdot}439/m$ microfarad in series with the line between two inductance leaks each of $1{\cdot}185/m$ henries; or (*c*) a contrivance every m miles consisting of two condensers in series, each of the capacity $0{\cdot}878/m$ microfarad, the point of junction being put to earth by an inductance of $0{\cdot}5925/m$ henry. It will be found that as m is smaller and smaller the line with detached contrivances approximates more and more to the line with continuous properties.

Detached Contrivances.

Suppose we have contrivances at the equidistant places A, B, &c., m miles apart. There is a contrivance whose terminals are A and A_0, another whose terminals are B and B_0, &c. Between A_0 and B we have the standard cable. Let the currents in the line at A, A_0, and B be c, c_0, and C. Let the voltages at these points be v, v_0, and V. Let $V/C=v/c=\rho$. Knowing the nature of the contrivance we can calculate v_0 and c_0 from v and c. It is also known that

$$V = v_0 \cosh mn + \frac{r+lqi}{n} c_0 \sinh mn,$$

$$C = c_0 \cosh mn + \frac{n}{r+lqi} v_0 \sinh mn,$$

where $$n = \sqrt{(r+lqi)(s+kqi)}.$$

Putting V/C or ρ equal to v/c, we have a quadratic to calculate ρ and therefore V and C.

I am in the habit of writing $c=1$, meaning $c = \sin qt$, and then $v=\rho$. Whatever the contrivance may be,

$$\frac{V}{C} = \rho = \frac{\alpha+\beta\rho}{a+b\rho},$$

where α, β, a and b are given in value; they are usually complex unreal quantities. Solving for ρ and finding C we have two answers, C_1 and C_2 say. In general $C_1C_2=1$, and if $\frac{1}{2}(a+\beta)$ be called P,

$$C = P \pm \sqrt{P^2-1}.$$

Now P is very easily evaluated, and therefore C.

Example :—(1) If our contrivance is a resistance r_1 in series with the line $c_0 = c = 1$, $r_0 = r_1 + \rho$. Thus if r_1 is an Ohmic resistance R in series with a capacity K and an inductance L,

$$r_1 = \mathrm{R} + \mathrm{L}qi + \frac{1}{\mathrm{K}qi}.$$

This is a very simple case, the simplest form of which is that of Mr. Pupin, a mere inductance coil in the line. The trouble in such a case is that the whole telephonic current passes through the coil and there is considerable Ohmic loss of energy. It is to be remembered that the value of R to be taken in any such case may be many times the ordinary resistance, because $c^2\mathrm{R}$ is taken to be the total loss of power, and this is due to hysteresis, eddy-currents, &c., as well as mere Ohmic loss.

(2) Again taking a case already mentioned; the contrivance consists of two equal resistances r_1 in series with the line and a resistance leak to earth r_3 from the point of junction of the other two.

It is easy to show that

$$c_0 = 1 + \frac{r_1}{r_3} + \frac{\rho}{r_3}$$

and

$$r_0 = 2r_1 + \frac{r_1^2}{r_3} + \rho\left(1 + \frac{r_1}{r_3}\right),$$

and it follows that

$$\mathrm{P} = \left(1 + \frac{r_1}{r_3}\right)\cosh mn + \left\{r_1\frac{n}{r}\left(1 + \frac{r_1}{2r_3}\right) + \frac{1}{2r_3}\frac{r}{n}\right\}\sinh mn.$$

It may be well to work out one example. In the above-mentioned cable $r = 18$, $k = 0{\cdot}055 \times 10^{-6}$, $s = 0$, $l = 0$; take $q = 5000$,

$$n = \sqrt{rkqi} = {\cdot}07036\ (45^\circ), \quad \frac{n}{r} = {\cdot}003909\ (45^\circ),$$

$$\frac{r}{n} = 255{\cdot}8\ (-45^\circ).$$

Let us place contrivances at $m = 4{\cdot}263$ miles apart. I choose this distance because $mn = 0{\cdot}3\ (45^\circ)$, and this happens to be given in Prof. Kennelly's table, which tells me that

$$\cosh mn = 1{\cdot}0007\ (2^\circ.\ 35'), \quad \sinh mn = 0{\cdot}29994\ (45^\circ.\ 51').$$

To imitate the continuous case given above we need an inductance leak of 0·5925 henry per mile or $L = \cdot5925 \div 4\cdot263$ every 4·263 miles. Hence

$$r_3 = Lqi = 694\cdot7\, i.$$

r_1 is a condenser of capacity $K = \frac{0\cdot878}{4\cdot263} 10^{-6}$, so that as $r_1 = \frac{1}{Kqi}$, $r_1 = -971\cdot13\, i$. Inserting these values in the above formula, I find

$$P = -0\cdot05384 - \cdot068078\, i$$

and

$$C = 1\cdot057\ (183^\circ.\ 54') \quad \text{or} \quad 0\cdot946\ (-183^\circ.\ 54').$$

That is, there is an attenuation or increase of 5·5 per cent. in 4·263 miles or 1·3 per cent. per mile. Observe how different this is from the effect of the continuous distribution.

As another case, a most important one, consider the standard subterranean cable $r = 88$, $k = \cdot05 \times 10^{-6}$, $s = 0$, $l = 0$.

I. Without contrivances $h = g = \sqrt{rkq/2}$. If $q = 5000$ (a frequency of 800), $h = \cdot1049 = g$. There is an attenuation of 11 per cent, per mile. It is to be observed that notes of frequencies 1 : 2 are attenuated in the proportion $1 : \sqrt{2}$. This is probably good, because it is known that the graver notes of speech need to be of greater intensity for distinct hearing. But the changes in phase are also as $1 : \sqrt{2}$, whereas for good hearing they ought to be as $\sqrt{2} : 1$.

Using this cable with no contrivances it is found in practice that there is good telephonic speech to a distance of 40 miles. When $q = 5000$ a current 1 is reduced to 0·015 or $\frac{1}{66}$ of itself in 40 miles. If $q = 6000$ (a frequency of 960) a current of 1 is reduced to 0·01 with a lag of 263°; whereas if $q = 4000$ (a frequency of 640) a current 1 is reduced only to 0·0238, but it has a lag of 215°. In fact the two lags in actual time are as 22 to 29. If, then, we find such attenuations and lags in any case we may infer that good telephone speaking is possible.

II. I will first give examples of a tuned circuit, that is, very perfect transmission of one musical note of frequency 800. Keeping to the contrivance which has two condensers in series, each of capacity K in the line and an inductance leak of L henries to earth from the point between the condensers, it is easy to write out the general expression for P in terms of m and r_1 and r_3 (at all events when mn is small), and it may be studied. Now for any particular m find the values of r_1 and r_3 which will cause P to be nearly 1 (0°) and therefore C to be nearly 1 (0°). I have already explained that we have no right with the above formulæ to get a really constant current like this, but we have a right to approach it very nearly.

If $m=1$ mile, then $K=\cdot5917$ microfarad and $L=\cdot8324$ henry. If $m=2\cdot755$ miles, $K=\cdot2273$ microfarad and $L=\cdot3692$ henry. Even when m is as much as 5·51 miles, a very considerable distance for such a cable, I find that I can use values of K and L which cause the attenuation to be comparatively small.

We see that by means of contrivances placed at considerable distances apart we can tune a long circuit so that it will transmit a current of one frequency, just as if the line were a mere Ohmic resistance.

I do not know whether practical men see great value in having this power. It may be important to inventors. It is well, however, to say that when we have such a line, unless the contrivances are rather close together, the circuit is really a tuned one, tuned to a particular frequency and not good for the transmission of other frequencies.

III. Isolated contrivances. When m is so great that $\cosh mn$ and $\sinh mn$ are practically equal, calculations are particularly easy. Thus, taking the case in which $q=5000$, $r=88$, $k=\cdot05\times10^{-6}$.

We have

$$\frac{v}{c}=\rho=\sqrt{r/kqi}=593\,(-45^\circ)=419\cdot3\,(1-i)$$
$$=\alpha-\alpha i=\alpha-\beta qi, \text{ say.}$$

(1) If in the middle of a very long line we merely insert an inductance leak to earth of L henry; $L=\frac{\alpha}{q}=\beta=\cdot11725$ gives the best effect. This multiplies the received current by 1·414 and causes a lead of 45°.

(2) An inductance of $L = \cdot 11725$ henry in series with the line and a leak to earth which is a condenser of capacity $1 \div \beta q^2$ farads (in the present case 0·532 microfarad); this causes the current to be multiplied by 1·414 and to get a lead of 225°.

(3) An inductance of 0·11725 henry in series with the line and no leak to earth. This multiplies the received current by 1·414 and it creates a lag of 45°.

IV. Returning to the contrivance consisting of two equal condensers in series with the line and an inductance to earth from the point between the condensers. For good telephonic speech it is evident that the inductance ought to be small, otherwise there is too much dependence on frequency; there is some tuning.

We want k'/l' to be small and nearly constant and $k'l'$ to be small also. We cannot effect both these objects, but we can obtain fairly good speaking by compromise. Thus, let $k' = -\cdot 05 \times 10^{-6}$ and $l' = -\cdot 128$, so that $K = \frac{1}{3\cdot 2}$ microfarad and $L = 0\cdot 4$ henry. Thus if m is 1 mile, $r_1 = -320\,i$ and $r_2 = 2000\,i$. I find that

$$\text{if} \qquad q = 5000, \quad C = 0\cdot 9704\ (24^\circ . 3'),$$

$$\text{if} \qquad q = 4000, \quad C = 0\cdot 9647\ (35^\circ).$$

That is, with a frequency 800 there is an attenuation of 3 per cent. and a lag of 24° per mile, whereas with a frequency of 640 there is an attenuation of 3½ per cent. per mile and a lag of 35°. We get practically the same results by the continuous formulæ.

Practical men will now see their way to easy calculation of the effect of placing any kind of contrivance in a circuit, and I need not give any more examples. It is to be remembered that condensers suitable for telephonic purposes are very cheap, a few shillings per microfarad, whereas inductances are expensive. Also, it has to be remembered that when the whole telephonic current passes through an induction coil, its resistance becomes of very serious importance, for the cost of coils of similar construction is proportional to the 3/2 power of L/R if L is the inductance and R the resistance of the coil.

With the permission of the Society I here append a table of the values of $\sinh r\,(45^\circ)$ and $\cosh r\,(45^\circ)$ to supplement that published by Kennelly. His r is 0, 0·1, 0·2, &c., and we need closer entries for the above kind of calculation. My table has been computed by Mr. J. J. Brookes.

r.	$\sinh r\,(45^\circ)$.	$\cosh r\,(45^\circ)$.
·01	·010000 (45°·00)	1·00000 (·00°)
·02	·020000 (45°·00)	1·00000 (·00°)
·03	·030000 (45°·00)	1·00000 (·02°)
·04	·040000 (45°·01)	1·00000 (·04°)
·05	·050000 (45°·02)	1·00000 (·07°)
·06	·060000 (45°·04)	1·00000 (·10°)
·07	·070000 (45°·05)	1·00000 (·14°)
·08	·080000 (45°·06)	1·00000 (·18°)
·09	·090000 (45°·08)	1·00000 (·22°)
·10	·10000 (45°·09)	1·00001 (·29°)
·15	·15000 (45°·21)	1·00004 (·64°)
·20	·20000 (45°·38)	1·00013 (1°·15)
·25	·25000 (45°·60)	1·00032 (1°·79)
·30	·30001 (45°·86)	1·00067 (2°·56)
·35	·35003 (46°·17)	1·00125 (3°·50)
·40	·40005 (46°·53)	1·0021 (4°·58)
·45	·45010 (46°·93)	1·0034 (5°·79)
·50	·50016 (47°·39)	1·0052 (7°·14)
·55	·55027 (47°·89)	1·0076 (8°·63)
·60	·60042 (48°·44)	1·0107 (10°·25)
·65	·65063 (49°·03)	1·0148 (12°·01)
·70	·70093 (49°·68)	1·0200 (13°·89)
·75	·75131 (50°·37)	1·0263 (15°·89)
·80	·80181 (51°·10)	1·0336 (18°·01)
·85	·85245 (51°·89)	1·0426 (20°·24)
·90	·90327 (52°·72)	1·0533 (22°·57)
·95	·95428 (53°·60)	1·0659 (24°·99)
1·00	1·00553 (54°·53)	1·0803 (27°·49)

ABSTRACT.

The Author published a paper in the Proceedings of the Society in 1893, showing how voltage v and current c are attenuated along a telephone or submarine telegraph-line, a line with resistance r, capacity k, inductance l, and leakance s per unit length; currents are of the form $\sin qt$. When lq/r is considerable the mathematical expressions become simple. It was pointed out that the introduction of l is of great benefit. The Author now points out that k may be made negative by the use of inductance leaks to earth, and l may be made negative by the use of condensers in series with the line. To introduce l as Mr. Pupin has done by inductance coils at equidistant places on the line, or to introduce the other properties mentioned by placing other contrivances at equal

distances, is a mathematical problem of great complexity. Contrivances placed close together have the same effect as the continuous distribution of properties, but there is considerable expense; the problem is to find how far apart the contrivances may be placed so that the effect produced shall still be beneficial. Mr. Pupin has given a rule for the spacing of his coils, but practical men dispute its accuracy; nobody has given a rule for other contrivances; the object of the Author is to give an easy method of calculation which is practically correct and which can be used when the contrivance is any network or other combination of resistances, inductances, and capacities—some being leaks to earth—and it may include transformers, motors, and generators.

Suppose there are contrivances at the equidistant places A, B, &c., m miles apart in a cable which has the above mentioned properties r, k, l and s. There is a contrivance whose terminals are A and A_0, another whose terminals are B and B_0; between A_0 and B there is m miles of cable. Let the currents in the line at A, A_0 and B be c, c_0, and C. Let the voltages at these points be v, v_0, and V. The assumption on which the whole method is based is that $V/C = v/c = \rho$. This is practically true everywhere in a long line except near the ends. Now whatever be the nature of the contrivance we can calculate v_0 and c_0 from v and c. It is also known that

$$V = v_0 \cosh mn + \frac{r + lqi}{n} c_0 \sinh mn,$$

$$C = c_0 \cosh mn + \frac{n}{r + lqi} v_0 \sinh mn,$$

when $$n = \sqrt{(r + lqi)(s + kqi)}.$$

Putting V/C or ρ equal to v/c we have a quadratic to calculate ρ and therefore V and C, and the problem is solved. Taking $c = \sin qt$ and calling it 1 then $v = \rho$. Whatever the contrivance may be we find that $V = \alpha + \beta\rho$ and $C = a + b\rho$, where α, β, a, and b are given in value; they are usually unreal quantities of the form $M + Ni$ where i is $\sqrt{-1}$. Solving for ρ and finding C we have two answers which are reciprocals of one another. If $\frac{1}{2}(\alpha + \beta)$ be called P, and this is very easily evaluated, then

$$C = P \pm \sqrt{P^2 - 1}.$$

Examples of the use of the method are given, some showing that the detached contrivances produce much the same and others very different effects from what might have been expected from a study of the cable with continuous properties. It was shown that a line may have contrivances somewhat far apart which will tune it to a musical note merely, so that it acts almost like an ohmic resistance, but which will not transmit well the currents of other frequencies, and that for the commercial transmission of speech there must be a compromise. The Author impressed on the audience the fact that his method of calculation could be taught to quite non-mathematical people.

Discussion.

Major O'Meara thanked Prof. Perry for his very interesting and useful paper, and wished to assure him on behalf of a very large body of Telephone Engineers that they much appreciated the assistance which he and other mathematicians had rendered by their mathematical statement of the Engineer's case. The equations given were of the same value to engineers as the compass was to the mariner. The only question they were in doubt about was whether all the forces that come into play had been equated. They consequently made a preliminary investigation of the manner in which the equations had been built up, and this convinced them that certainly the most important of the forces had been introduced into the equation. Some of them were responsible for the expenditure of large sums of public money, and the fact that they had something reliable to guide them in making their recommendations had removed a considerable amount of anxiety. As a matter of fact, at the Post Office they had for some time past actually applied the formulæ, and had obtained a considerable amount of success in doing so, so far as the introduction of loading coils was concerned. For instance, a short while back the question of a new cable between England and France was under consideration. It had been stated in foreign technical journals that the formulæ for "loaded" cables did not apply to the case of conductors covered with gutta-percha, owing to the very large electrostatic capacity introduced and the peculiarities of the insulation. Although he (Major O'Meara) had faith in these formulæ, still he felt that a practical experiment would remove all doubt, and in consequence he arranged an experiment with some old No. 7 gutta-percha wire, *i. e.* copper wire weighing 40 lbs. per mile covered with 50 lbs. of gutta-percha per mile, which had a resistance per loop mile of 44 ohms, and an electrostatic capacity wire to wire of 0·13 microfarad per mile.

The gutta-percha wire was immersed in a tank, and experiments were conducted on lengths varying from 15 to 105 miles. They did not design special coils, but made use of some already in stock. These coils had an inductance of 83 millihenries, and a resistance of 13·4 ohms at 750 p.p.s. Admittedly the coils were not the best for loading this particular type of cable.

Having decided to place these coils at 1½ mile intervals, that is, to insert 55 m.h. per mile, they calculated the attenuation constant which would result. It worked out at 0·0427 per mile. When the coils were inserted, and the experiment completed, the observed attenuation constant was found to be 0·0419 per mile. From this it was evident that the calculated and observed results practically agreed.

The information thus obtained enabled them to deal with the question of the new submarine telephone cable, and they decided that the attenuation constant for this cable should be 0·0147 per mile, to be obtained by the insertion of coils at one nautical mile apart. The coils have a resistance of slightly less than 6 ohms, and possess an inductance of 0·100 henry at 750 periods per second.

The cable has been manufactured, and tested; the actual observed attenuation constant is 0·0140 per mile, so that this again proved the value of the formulæ.

The new telephone cable consists of 160 lbs. copper wire, and 300 lbs. gutta-percha. It has a conductor resistance per loop mile of 12·5 ohms (excluding the coils) and a wire to wire capacity of 0·12 microfarads per mile. In an unloaded state, the attenuation constant of this cable is 0·045 per mile.

Major O'Meara mentioned that the question of leakance was certainly of very great importance. It was owing to the very large part that leakance played that their first experiments were not successful. In America they had commenced to load aerial wires, and so large a part does this leakance play that in that country telephone engineers had been obliged to abandon the use of glass insulators, and substitute the highest grade of porcelain insulators to meet the difficulties which had been met with owing to an insufficient allowance for the part this factor played. He feared that in no case could leakance be neglected, whether aerial or underground conductors were used.

Mr. A. W. **Martin** said that most telephone engineers in this country and America considered that a circuit should be capable of transmitting currents of frequencies varying from 100 to 2000 p.p.s. in order that good clear speech might result. For attenuation calculations, some take 750, and others 800 p.p.s. as a mean value. Where lq and kq are great compared with r and s respectively,

$$h=\frac{1}{2}\left(r+\frac{sl}{k}\right)\sqrt{\frac{k}{l}},$$

and writing $s=0$,

$$h=\frac{r}{2}\sqrt{\frac{k}{l}}.$$

The question is not whether $s=0$, but whether the ratio of s to k may be regarded as zero; since both are small quantities. In practice $\frac{s}{k}$ is found to be far from negligible when inductance is placed in the circuit. When inductance L with resistance R is placed in leak, and assuming R small relatively to Lq, the impedance takes the place of ordinary insulation resistance and is practically equal to Lq. The transmission efficiency of the whole circuit for low telephonic frequencies would be low, and there would be an appreciable C^2R loss in the leak coils. When lq is not great compared with r but where the insulation resistance is much above a megohm per mile, s may be neglected without causing calculated and observed results to differ materially. The expression then becomes

$$\sqrt{\tfrac{1}{2}qk(\sqrt{r^2+q^2l^2}\pm ql)}=g\,(+)\text{ or }h\,(-).$$

If r be high compared with ql no great error is made by writing

$$g=h=\sqrt{qkr/2},$$

but as l in cable loops is never less than 0·001, it cannot be neglected except, say, when r exceeds 50 ohms. The difference between the values of q and h becomes great as r becomes small. A circuit tuned to a particular frequency would be useless for telephone working but might prove of value telegraphically. The ordinary paper and wax insulated, thin lead-foil plate condenser offers resistance to the transmission of high frequency currents, and the insertion of such condensers in a telephone loop would be accompanied by a C^2R loss. Experimental results show that improvements of 370 per cent. by coil loading (Pupin method) may be obtained in telephone loops. The 18 ohm per mile cable mentioned in the paper had a measured attenuation constant of 0·045 per mile before loading. After the insertion of certain coils at 3·2 mile intervals, each coil having an inductance of 0·133 henry and offering an effective resistance of 5·4 ohms at 750 p.p.s., the observed attenuation constant per mile was 0·012, which result is fairly in agreement with the value of 0·013 obtained by calculation at 750 p.p.s. from

$$h=\frac{1}{2}\left(r+\frac{sl}{k}\right)\sqrt{\frac{k}{l}} \quad \text{where } \frac{s}{k}=80.$$

When the coils were placed at 2·1 mile intervals, calculation and observation gave the attenuation constant as 0·012 per mile. Coils with cores of properly chosen magnetic material give a higher inductance for a given effective resistance than air-core coils, and are therefore the better type. Experience shows that the quality of speech transmitted becomes bad when the number of coils per wave-length at 2000 p.p.s. is less than π.

Measurements taken on various lengths of cable, the copper conductors in which were continuously and closely wound over with fine iron wire, showed that this "continuous" form of loading produces an improvement of only 60 per cent. upon the untreated condition in cases where the copper is of 2 sq. mm. area and upwards. With smaller conductors it may possibly reach 100 per cent.

Mr. B. S. **Cohen** said that the impressions obtained after reading this paper were that Prof. Perry had assumed that a telephone circuit tuned to the average telephonic frequency would as a result of this tuning transmit speech-waves more efficiently. He thought, however, that those who had studied this subject would agree that this was a misconception. He (Mr. Cohen) had previously pointed out that an average speech-wave consisted of a fundamental varying between 100∼ and 300∼, the highest harmonic being about 1500∼ and the average ∼ of the whole wave 800∼. He had also given the Fourier series obtained by analysing a number of speech-waves, and this series showed that there are a large number of odd and even harmonics of approximately equal importance ranging from the 2nd to the 11th harmonic.

This would in itself explain why it was not only useless but absolutely harmful to tune to any particular frequency. Turning to the 18 ohm cable line referred to by Prof. Perry, he gave some results worked out by Mr. G. M. Shepherd. When this cable is in the normal condition the attenuation for a wave with a value q (angular velocity) of 1000 is 2·2 per cent., for a value of 5000 it is 3·1 per cent., and for 9000 6·4 per cent.

This covered the whole commercial speech range, so that the variation over this range for attenuation is about 300 per cent. Using Prof. Perry's formula and taking his contrivances adding ·593 henry per mile and a capacity of ·44 mfd. in series, it will be found that the variation of attenuation between $q=1000$ and $q=5000$ is 500 per cent. That is to say, up to the average frequency alone the variation in attenuation is 200 per cent. more than over the whole range of speech frequency with the normal cable. But going a step further for an angular velocity of 5550, the attenuation constant is 0 and the current at the end of the line using the formula given is 0. This means that the inductance and capacity bridge act as a shunt of infinitely low resistance. Any further increase in the frequency results in a negative attenuation, implying that an entire change in the nature of the transmission takes place. It would be of much interest if Prof. Perry would explain exactly what is the physical interpretation of this curious effect. Whatever the explanation, however, there is little doubt that the effect of this combination of bridges L and K would be satisfactory, as far as the commercial transmission is concerned, in spite of the fact that at 800 ~ the attenuation is only 33 per cent., or an improvement for that particular frequency alone of about 15 times over the normal line. Contrasting these results with those which would be obtained by the application of the ordinary Pupin series loading coils of ·13 henry each spaced at 1-mile intervals in the same line, the attenuation at any frequency over the range of telephonic speech is 1·1 per cent.: that is to say, the volume improvement is 5, and the line acts as a distortionless one instead of amplifying the natural distortion of the unloaded line. He mentioned that this was not only a calculated result, but one that was amply confirmed by everyday practice, and which included the effect of the effective resistance of the inductance. Since his paper at the St. Louis Congress Prof. Kennelly had written a considerable number of papers on the subject of telephonic transmission, and his formulæ and methods were now adopted by many telephone engineers and were rapidly becoming standardized.

Dr. **Russell** thanked Professor Perry for his instructive paper. He much appreciated the methods shown of getting simple approximate formulæ for some very difficult problems. It was interesting to compare the Author's solution in the case when the leakance was zero with Heaviside's solution for a distortionless circuit. The only difference in the formulæ was that the attenuation was twice as great in the latter case. The Physical Society was deeply indebted to Major O'Meara and the other speakers for the very interesting data they had given about Pupinised telephone-lines. From the theoretical point of view it was exceedingly satisfactory to find that the "loading coils" fulfilled their functions so efficiently and that their performance could be predicted so accurately.

Dr. J. A. **Fleming** forwarded the following communication:—The subject of Prof. Perry's paper has an interest for me at the present moment, as I happen to be giving a course of Post-graduate lectures at University College on the propagation of currents in telephone and telegraph cables. The method of representing the quantities concerned by complex quantities and their hyperbolic functions is of course very familiar to all who have paid any attention to the subject, and this method, aided by Mr. Blakesley's Tables of Hyperbolic Functions, enables arithmetic calculations to be easily made. Whilst it is quite true that the wave-form of the currents transmitted in telephony is very far from being simple harmonic, yet experiment shows that when using sine-curve currents having a frequency of about 800 or an angular velocity of 5000, the results enable us to predict the effects with actual speech fairly well. The quantity it is important to predetermine is the current at the receiving end of a line of known length l and known constants, resistance, inductance, and capacity (R, L, C) per mile, when a receiver of known impedance $z'=R'+jpL'$ is placed at the end.

The important quantities are: the attenuation constant

$$\alpha=\sqrt{\frac{Cp}{2}[\sqrt{R^2+p^2L^2}-Lp]},$$

the wave-length

$$\lambda=2\pi/\beta=2\pi\Big/\sqrt{\frac{Cp}{2}[\sqrt{R^2+p^2L^2}+Lp]},$$

and the initial sending end impedance

$$z_0 = R+jpL/\alpha+j\beta.$$

If Pl (where $P=\alpha+j\beta$) is called the propagative length, and if

$$z'/z_0 = \tanh\gamma,$$

then it is quite easy to show that the ratio of the currents at the receiving and sending ends is given by

$$I_2/I_1 = \cosh\gamma \operatorname{sech}(Pl+\gamma).$$

Also that

$$I_1 = \frac{V_1}{z_0}\coth(Pl+\gamma),$$

where V_1 is the sending voltage. These formulæ are compact and easily applied in numerical calculations.

The problem with which Prof. Perry is dealing is, however, that of the loaded cable. He does not make any reference to the important paper of G. A. Campbell (see Phil. Mag. March 1903) or to that of Hayes (see 'Electrician,' Dec. 16, 1904), in which this problem is discussed. Campbell has given the following formula for the average or effective propagation constant $P'=\alpha'+j\beta'$ for a line consisting of impedance coils of impedance z' spaced out at intervals d on a line of propagation constant P and line impedance z_0:—

$$\cosh P'd = \cosh Pd+\frac{z^1}{2z_0}\sinh Pd;$$

but it is not certain that this formula can be trusted to give the true attenuation constant a' of the loaded line when d is much greater than about 1/8th of the wave-length on the loaded line, and when it is less than $\lambda/8$ then the attenuation constant can more easily be calculated by considering the added inductance and resistance smoothly distributed. Professor Perry does not discuss the question of the loss of amplitude due to reflexion from the interpolated impedances when these are spaced out to large fractions of a wave-length, nor the advantages of tapering off the loading at the ends of the loaded cable to reduce the effects of these reflexion losses at the ends. As an illustration of the general accuracy of the formula for the current ratios I may give the following figures, obtained for me by the kindness of Mr. Gill, Engineer-in-Chief of the National Telephone Company, in their Research Laboratory, by measurements made by Mr. Cohen. A standard line R = 88 ohms, C = ·054 mfd., L = ·001 henry, all per loop mile, had a receiving instrument of impedance 660|66° 54′ vector ohms placed at a distance of 15 miles. The ratio of the sending to the received current by the above formula was found by calculation to be 5·36, and by measurement with Mr. Cohen's barretter to be 5·3, thus showing a very good agreement. The frequency in this case is 1000 ∼. By the use of artificial loaded lines it is easy to show experimentally the enormous decrease in attenuation caused by properly loading the line. Whilst we owe to Heaviside the first insistance on the value of the added inductance, it is to Pupin we are more particularly indebted for proving that when impedance is spaced out at distances not greater than about $\lambda/8$, we have all the effects of smoothly distributed inductance but obtained more easily and with less expense.

Mr. G. M. SHEPHERD, in a letter to the Secretary, observed that the system of introducing inductive leaks referred to in Prof. Perry's paper, was presumably identical with what is termed "Thompson's" Compensated line. It could be shown that such a line was equivalent (proper spacing of loads assumed) to an ideal uniform line whose constants are

$$L' = L - \frac{2}{q^2 cs}\,\frac{(q^2 L_0 L + R_0 R)}{(q^2 L_0^2 + R_0^2)},$$

$$R' = R - \frac{2}{cs}\,\frac{(L_0 R - L R_0)}{(q^2 L_0^2 + R_0^2)},$$

where L, C, R are the ordinary wire constants, and L_0, R_0 the inductance and resistance of the inductive leaks, s being the spacing. It could also be shown that the rule governing the spacing was virtually the same as that formulated by Pupin for series coils. Looking at the matter from this point of view, it is seen that the effect of loading on the Thompson system is to reduce line resistance and inductance, or the inverse to the Pupin line. The two systems, however, appear to be essentially different, in that the Thompson line is really tuned or compensated, while the Pupin line is not. Hence the former is more or less subject to frequency, but the latter practically independent of it. A few cases worked out from the above formula for 18 ω, ·055 mfd. cable, with a ·972 henry leak per mile, and several frequencies not far apart, give the following figures:—

~	q.	h.
1035	6500	·0189
1115	7000	·0281
1430	9000	·0487

This shows a wide departure from a distortionless line. Again, introducing series capacity brings in a negative inductance which is likewise a function of frequency, so that a composite contrivance of the type indicated by Prof. Perry was likely to be a very fickle aid to telephony, a conclusion fully borne out by practical tests. The insertion of series inductance originally advocated by Heaviside, and developed by Pupin, seemed to be the only natural solution for the telephone problem, and recent experience confirmed this view.

Prof. PERRY communicated the following reply:—"I am glad to think that my paper has led to Major O'Meara's remarks. The results of the Post Office experiments have been freely put before us by Mr. Martin; they are of enormous value to people like myself who desire to make calculations. Such experiments are quite impossible for the ordinary student, they cost a great deal of money and the service of many assistants. I am very glad to think that these results will be published in full by the Society. I regret that Mr. Cohen did not speak at much greater length, for he has for a long time been experimenting on behalf of the National Telephone Company and the results of his experiments would be of great value to students.

"I am afraid, however, that my paper has been a little misunderstood by these and other critics. They seemed to think that I was advocating the use of a contrivance consisting of condensers in the line and an inductance leak, whereas I mentioned this contrivance merely to illustrate my paper. What I have endeavoured to do is to show how calculations may be made by non-mathematical people to find the effect of using any contrivance whatsoever. The above abstract does not, I am sorry to say, indicate my praise of the work done by Prof. Kennelly, who by his many papers and his valuable tables of cosh and sinh functions has cleared up many difficulties. It is enough here to say that except the few lines devoted to Mr. Campbell's paper, the whole of Dr. Fleming's communication will be found to have been published by Prof. Kennelly many years ago."

XVIII. *On the Laws regarding the Direction of Thermo-electric Currents enunciated by M. Thomas. By* CHARLES H. LEES, *D.Sc., F.R.S., Professor of Physics in the East London College, University of London*[*].

IN number 8 of the *Bulletin de la Classe des Sciences* of the Académie Royale de Belgique for 1909, M. Bruno-Joseph Thomas enunciates the following laws as the result of his examination of the direction of the electric current produced in more than 100 thermo-couples :—

"1. When a difference of temperature is established between the two junctions of a thermo-electric couple, the current produced flows from the hotter to the colder junction through the metal which is electrically the better conductor, if the product of the coefficient of electrical conductivity of that metal by the coefficient of thermal conductivity of the other metal exceeds the product of the coefficient of thermal conductivity of the first metal by the coefficient of electrical conductivity of the second.

"2. If the former product is less than the second, the current flows from the hotter to the colder junction through the metal electrically the worse conductor.

"3. If the two products are equal no current is produced."[†]

The author states further that at different temperatures "for a given couple the strength of the current produced increases with the ratio ("rapport") of these two products, and other things remaining equal, the strength will be proportional to that ratio"[‡].

M. Thomas supports his theory by 14 pages of tables, in which the values of the products for each of the 105 couples which can be obtained by combining together in pairs the 15 metals of Becquerel's thermo-electric series, are compared. Not a single exception to the theory is found, and at first sight it appears as if M. Thomas had succeeded in establishing a most important and hitherto unsuspected law of connexion between the thermo-electric properties of a metal and its conductivity for heat and for electricity. On more careful examination, however, it is seen that M. Thomas in his tabular proof has based his argument entirely on the order of the metals in Becquerel's thermo-electric series, on the heat conductivities of 7 of the metals according to Berget, of 2 others according to Wiedemann and Franz, and on the electrical conductivities of 12 of them according to Latimer Clark, and of 1 other according to Becquerel. Of the heat conductivities of the 9 metals thus obtained M. Thomas states himself that he modifies 4 by amounts varying from 2 to 10 per cent. in order to make the values of the products fit in with his laws *. As he had no values for the heat conductivities of the remaining 6 metals, nor for the electrical conductivities of 3 of the metals, he states himself that he assumes values for these quantities which fit in with his law †.

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[*] Read February 25, 1910.

[†] *Bulletin, Classe des Sciences, Roy. Acad. Belg.* 1909, p. 903.

[‡] *Ibid.* p. 925.

It cannot be claimed that such a method of treatment provides a firm support for the theory advanced by M. Thomas, and it is necessary to examine it more critically in the light of the more accurate values of the constants which are available.

In the first place, it is possible to put the theory into a more compact form than that adopted by M. Thomas. According to him, if electrical conductivity of the better electrical conductor × heat conductivity of the worse electrical conductor is greater than electrical conductivity of the worse electrical conductor × heat conductivity of better electrical conductor the thermo-current flows from hot to cold in the better electrical conductor.

If, on the other hand, electrical conductivity of better electrical conductor × heat conductivity of worse electrical conductor is less than electrical conductivity of worse elec-

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* With regard to the values for these four metals—copper, gold, iron, and lead—M. Thomas says (p. 904):—"Si je les acceptais tels qu'ils sont, j'aurais nécessairement un petit nombre de couples dont le sens du courant ne répondrait pas à mes lois. Je les ai donc modifiés légèrement ainsi qu'il suit:—"

† Of the latter quantities M. Thomas says (p. 906):—"Je ne connais ni les coefficients de conductibilité électrique, ni les coefficients de conductibilité calorifique du cobalt, du manganèse et de l'arsenic, ... Je prendrai donc par hypothèse pour coefficients du conductibilité électrique du cobalt, du manganèse, et de l'arsenic les nombres 1380, 1000, et 200."

trical conductor × heat conductivity of better electrical conductor the thermo-current flows from hot to cold in the worse electrical conductor.

Dividing the inequalities through by the product of the heat conductivities of the two metals, the statements become:—

If $\dfrac{\text{electrical conductivity}}{\text{heat conductivity}}$ of better electrical conductor $\gtrless \dfrac{\text{electrical conductivity}}{\text{heat conductivity}}$ of worse electrical conductor the thermo-current flows from hot to cold in the $\left\{\begin{matrix}\text{better}\\ \text{worse}\end{matrix}\right\}$ electrical conductor. That is, the current flows from hot to cold in the conductor for which $\dfrac{\text{electrical conductivity}}{\text{heat conductivity}}$ is the greater.

In the original form the law was tested by a comparison of the 210 products of the electrical and heat conductivities of pairs of the 15 metals; in its present form it may be tested by a comparison of the 15 quotients of the electrical by the heat conductivities of the 15 metals.

It may be noted that the electronic theories of conduction of heat and electricity in metals make this quotient the same for all metals, and that these theories and that of M. Thomas are therefore incompatible.

In selecting the experimental material with which to test M. Thomas's theory it is important to remember that both the electrical and the thermal conductivities and the thermo-electric properties of metals are influenced by the presence of impurities, and that it is advisable to use only results obtained with pure materials. If, however, results for absolutely pure materials are not available, those for approximately pure materials may be used, if in the case of any particular metal the impurities in the samples used in the different sets of observations were the same in amount and in material, or better if the whole of the measurements have been made on the same material. Fortunately in the present case we have the experiments of Jäger and Diesselhorst *, who determined the quotient of the thermal by the electrical

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* W. Jäger and H. Diesselhorst, *Wissenschaftl. Abhandl. der Phys.-Techn.-Reichsanstalt*, iii. p. 270 (1900).

conductivity and the thermo-electric power of each of a number of rods of very nearly pure metals. Their results for the quotients of the two conductivities have been confirmed by more recent observations made by other methods [*], while their results for the thermo-electric powers of the metals agree with those obtained previously for approximately pure metals [†].

They may be used therefore without hesitation as the best results available at the present time for the comparison of M. Thomas's theory with experiment.

The following table contains their results at 18° C. for each of the metals tested. Where they give more than one result that one has been used to which they attach most weight. The unit of heat is the watt second; electrical conductivity is expressed in reciprocal ohms per centimetre cube :—

Metal.	Conductivities,		Thermo-electric powers in micro-volts per degree; with respect to	
	$\frac{\text{thermal}}{\text{electrical}}$.	$\frac{\text{electrical}}{\text{thermal}}$.	copper A.	lead.
Aluminium	·636 $\times 10^{-5}$	1·57 $\times 10^{5}$	+3·2	−0·3
Copper	·665	1·50	−0·3	+3·2
Zinc	·672	1·49	−0·1	+3·0
Silver..............	·686	1·46	0	+2·9
Nickel	·699	1·43	+22·3	−19·4
Cadmium	·706	1·42	−0·6	+3·5
Gold	·709	1·41	0	+2·9
Lead	·715	1·40	+2·9	0
Tin..................	·735	1·36	+2·6	+0·3
Platinum	·753	1·33	+6·1	−3·2
Palladium.........	·754	1·326	+11·5	−8·6
Iron	·838	1·19	−8·4	+11·3
Bismuth	·964	1·04	+73·3	−70·4

The numbers in the second and fourth columns are those given by Jäger and Diesselhorst, the third column contains the reciprocals of those in the second, the fourth gives the thermo-electric powers of the metals with respect to one of the copper bars, the fifth has been calculated from the fourth and gives the thermo-electric powers with respect to lead. The signs have been reversed to comply with the convention which makes the thermo-electric power of a metal positive if the current flows from lead to the metal through the hot junction, *i. e.* if the current flows from hot to cold through the metal.

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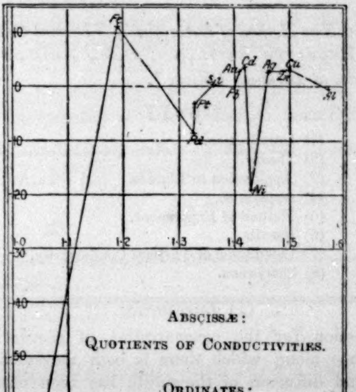
[*] C. H. Lees, Phil. Trans. Roy. Soc. Lond. A. 208, p. 381 (1908).

[†] K. Noll, *Ann. der Physik*, liii. p. 874 (1894); W. H. Steele, Phil. Mag. xxxvii. p. 218 (1894).

If the laws stated by M. Thomas are true the numbers in the third and fifth columns of the table should increase and decrease together. It is seen that as the numbers in the third column decrease those in the fifth change irregularly in magnitude and are sometimes positive, sometimes negative. They afford therefore no confirmation of the laws, but on the contrary disprove them.

The following diagram, in which each metal is represented by a point, with abscissa the quotient of the electrical by the thermal conductivity and ordinate the thermo-electric power of the metal, shows perhaps more clearly than the table the



want of agreement of the theory with fact. On the theory the current flows from hot to cold in that metal of a pair whose abscissa is the greater. According to the observations the current flows from hot to cold in that metal of a pair whose ordinate is the greater. The two are only in agreement when the line on the diagram joining the two points corresponding to any two metals slopes upward to the right. It is obvious that the number of exceptions to this rule, even for the metals dealt with, must be almost as large as the number of agreements, and that the laws enunciated by M. Thomas are not supported by the observed facts.

XIX. *On a New Method of Determining Thermal Conductivity.* *By* H. Redmayne Nettleton, *B.Sc., Assistant Lecturer in Physics at Birkbeck College* [*].

[Plate V.]

(1) Introduction.
(2) Theory.
(3) Application to Liquids.
(4) Apparatus.
(5) Method of Experiment.
(6) Results.
(7) Calculation of Thermal Conductivity.
(8) Conclusion.

(1) *Introduction.*

An expression for the concentration of a solution in a vertical tube along which there is both movement of the solution and diffusion of the solute has been obtained by Dr. A. Griffiths (Phil. Mag. Nov. 1898), who pointed out to the author that a similar expression could be obtained in the case of thermal diffusion and suggested the present investigation, the object of which is :—

1st. To find the effect of impressed velocity on temperature gradient.

2nd. To test the practicability of using such artificially produced temperature gradients as a means of determining thermal conductivity in absolute measure.

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[*] Read February 25, 1910.

The author has extended Dr. Griffiths's results so as to take into account radiation, which plays an important part in most cases of conduction, and has carried out the researches suggested.

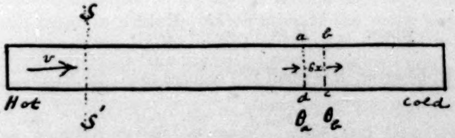
(2) *Theory.*

Consider a bar moving longitudinally with a constant velocity v as shown and passing through two constant temperature sources, and let

K = thermal conductivity,
s = specific heat,
ρ = density,
A = cross-section,
I = intrinsic heat in volume vA of the bar at the temperature of the enclosure,
θ = temperature above the enclosure at distance x from any plane in space normal to the bar.

Then the quantity of heat crossing a section SS′ in space is due to conductivity $-KA\frac{d\theta}{dx}$, and due to the impressed velocity $vA\rho s\theta + I$.

Fig. 1.



Applying this to the small prism $a\,b\,c\,d$ in space of width δx we have :—

$$\text{Heat entering } ad \text{ per sec.} = -KA\frac{d\theta_a}{dx} + vA\rho s\theta_a + I.$$

$$\text{Heat leaving } bc \text{ per sec.} = -KA\frac{d\theta_b}{dx} + vA\rho s\theta_b + I.$$

In the steady state the difference will be radiated, and since

$$\theta_b = \theta_a + \delta x\frac{d\theta}{dx},$$

this difference is given by

$$\mathrm{KA}\delta x\frac{d^2\theta}{dx^2}-v\mathrm{A}\rho s\delta x\frac{d\theta}{dx}.$$

But assuming Newton's Law of Emission and reckoning all temperatures from the room taken as zero, we may also express the heat radiated by the section *abcd* by the expression $\mathrm{E}p\theta\delta x$, where p is the perimeter of the bar and E the coefficient of emissivity. Equating we obtain :—

$$\mathrm{KA}\frac{d^2\theta}{dx^2}-v\mathrm{A}\rho s\frac{d\theta}{dx}=\mathrm{E}p\theta.$$

The general solution of this differential equation is:—

$$\theta=e^{\frac{v\rho sx}{2\mathrm{K}}}\left(\mathrm{A}e^{\frac{x}{2}\sqrt{\frac{v^2\rho^2s^2}{\mathrm{K}^2}+\frac{4\mathrm{E}p}{\mathrm{KA}}}}+\mathrm{B}e^{-\frac{x}{2}\sqrt{\frac{v^2\rho^2s^2}{\mathrm{K}^2}+\frac{4\mathrm{E}p}{\mathrm{KA}}}}\right),$$

where A and B are constants of integration.

Taking now θ_1, θ_m, θ_2 as the temperatures above the room at the positions $x=0$; $x=\frac{\mathrm{L}}{2}$; $x=\mathrm{L}$ respectively, we obtain

$$\frac{\theta_2-e^{\frac{v\rho s\mathrm{L}}{2\mathrm{K}}}e^{-\sqrt{\mathrm{R}}}\theta_1}{\theta_m-e^{\frac{v\rho s\mathrm{L}}{4\mathrm{K}}}e^{-\frac{1}{2}\sqrt{\mathrm{R}}}\theta_1}=e^{\frac{v\rho s\mathrm{L}}{4\mathrm{K}}}e^{-\frac{1}{2}\sqrt{\mathrm{R}}}\left(e^{\sqrt{\mathrm{R}}}+1\right)\quad .\quad .\quad (1)$$

where $\mathrm{R}=\frac{v^2\rho^2s^2\mathrm{L}^2}{4\mathrm{K}^2}+\frac{\mathrm{E}p\mathrm{L}^2}{\mathrm{KA}}.$

If, however, E is small this reduces to

$$\frac{\theta_2-e^{\frac{-\mathrm{E}p\mathrm{L}}{2v\rho s\mathrm{A}}}\theta_m}{\theta_m-e^{\frac{-\mathrm{E}p\mathrm{L}}{2v\rho s\mathrm{A}}}\theta_1}=e^{\frac{v\rho s\mathrm{L}}{2\mathrm{K}}+\frac{\mathrm{E}p\mathrm{L}}{2v\rho s\mathrm{A}}},\quad .\quad .\quad .\quad .\quad (2)$$

and if E could be neglected we should have

$$\frac{\theta_2-\theta_m}{\theta_m-\theta_1}=e^{\frac{v\rho s\mathrm{L}}{2\mathrm{K}}}.\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad (3)$$

It is hoped that with the aid of high-vacuum jacketed vessels equations (2) and (3) will be found useful when applied to liquids, but from the present point of view a particular case of equation (1) is of more importance.

Suppose θ_m = temperature of room = zero.

Then, whatever be the value of E, we obtain :—

$$\frac{-\theta_2}{\theta_1} = e^{\frac{v\rho s\mathrm{L}}{2\mathrm{K}}}$$

or

$$\mathrm{K} = \frac{\frac{1}{2}v\rho s\mathrm{L}}{\log_e \frac{-\theta_2}{\theta_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the ratio of the temperatures above and below—at equal distances from the point in the bar which is at the same temperature as the enclosure—is independent of the value of the emissivity.

Equation (4) is used below for calculating the value of the thermal conductivity of mercury.

(3) *Application to Liquids.*

There are experimental difficulties, probably not insuperable, in determining the temperature at a given point fixed in space and situate within a moving solid, and it was thought advisable to experiment in the first place with a moving column of mercury.

When a column of mercury at a constant temperature moves slowly along a tube, the velocity is a maximum at the centre and a minimum at the circumference, viscosity being the main factor in determining the form of the movement. When, however, the top of such a column is heated and the lower end kept cold, the velocity of the mercury at the central part of the tube being greater than near the walls, the average temperature of the central parts is less than the average temperature of the outside parts, and therefore the average density of the central parts is greater than the average density of the outside parts. Thus the gravity head in the central part is greater than the gravity head elsewhere, and this tends to diminish the differences in velocity at various parts of the cross-section. A mathematical investigation leads to differential equations of an exceedingly complicated type, but the author is of opinion that, neglecting radial conductivity, the isothermal surfaces would be approximately horizontal to quite a considerable distance from the centre of the tube.

It is not unlikely, however, that radial conductivity will play an important part, and if the diameter of the tube is small in comparison with the length, and if the flow is small, the variations in temperature radially will undoubtedly be small in comparison with the longitudinal variations. If the radial conductivity were infinitely great, or if the diameter were exceedingly small, the isothermals would be horizontal and the gravity effect alluded to above would vanish and, in the opinion of the author, the flow would again approximate to the well known parabolic character.

So long, however, as the isothermals are approximately horizontal the nature of the flow does not affect the result:—

For let $v\rho$ = the vertical component of the product of the velocity and density at a distance r from the axis ;

and let Q = quantity of heat crossing a horizontal plane per sec.

Then
$$Q = -KA\frac{d\theta}{dx} + \int_0^R v\rho s\theta \times 2\pi r \,.\, dr + I.$$

But
$$\begin{aligned}\int_0^R v\rho s\theta \times 2\pi r \,.\, dr &= s\theta\int_0^R 2\pi r \times v\rho \,.\, dr\\ &= s\theta \times \text{mass crossing section}\\ &= s\theta \times v'\rho' A \qquad [\text{per sec.}\end{aligned}$$

where $v'\rho'$ may be called the average product of velocity and density. This, by the law of fluid continuity, is constant longitudinally, so the method will not be affected by variations of density in the parts of the liquid at different temperatures. We have then :—

$$Q = -KA\frac{d\theta}{dx} + v'\rho' As\theta + I$$

and hence, just as in section (2), we obtain under conditions similar to those holding in equation (4)

$$K = \frac{\frac{1}{2}v'\rho' sL}{\log_e \frac{-\theta_2}{\theta_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

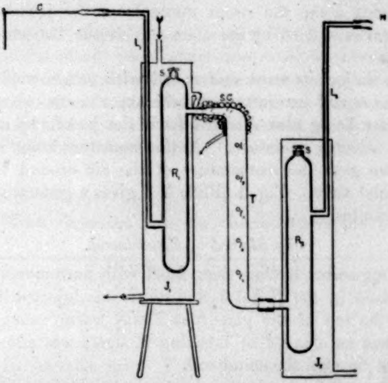
the product $v'\rho'$ being equal to the mass of liquid outflowing per sec. divided by the cross-section of the tube.

(4) *Apparatus.*

Preliminary experiments showed the importance and the difficulty of maintaining the ends of the experimental tube at constant temperatures in spite of the inflow of mercury.

The form eventually adopted consists of two specially cast iron reservoirs R_1 R_2 (fig. 2) about 40 cms. long and of 5 cms. internal diameter, to which by strong rubber tubing bound by wire the glass tube T was attached. This latter a little less than 3 cms. in diameter and 40 cms. long—its ends being bent as shown—had five side limbs through which thermometers $t_1 - t_5$ were inserted and fastened in with cement. The thermometers had cylindrical bulbs of small radius and were about 6 cms. apart.

Fig. 2.



Two limbs L_1 L_2 were attached to the iron reservoirs and these latter were surrounded with tin-jackets J_1 J_2. Screws S_1 and S_2 were provided so as to allow the air to escape in filling the apparatus, these screws being inserted with the aid of cement as soon as the mercury reached the top of the reservoirs.

A capillary tube C was attached to the top of the limb L_1, and to L_2 pressure tubing was attached leading to a vessel in which the head of mercury was kept constant by an overflow pipe. By raising or lowering this vessel the head could be varied and a constant outflow of mercury from the capillary C could be obtained.

A spiral SC of flexible tin tubing leaving the steam jacket J_1 was found useful for keeping up the temperature, thus enabling a larger temperature-slope to be investigated; for by turning a tap the steam would leave the jacket J_1 by the spiral as well as by the main tap at the bottom of the jacket.

Both tin jackets were surrounded with cotton-wool which was also bound carefully round the experimental glass tube, the latter being also shielded from the jackets by a thick-walled asbestos enclosure. A thermometer hung in this enclosure gave the temperature of the air around the experimental tube. Fig. 1 (Plate V.) gives a general view of the apparatus.

(5) *Method of Experiment.*

The apparatus having been filled with pure mercury, ice was placed in the jacket J_2 and steam was injected into J_1. When the top of the glass tube T was warm enough and there was no danger of breaking it, steam was allowed to circulate through the spiral coil.

In about 3 hours the steady state was obtained and the temperatures were noted. The head H was then raised till a suitable flow up the tube T and out of the capillary resulted. The weight flowing out in a definite time—usually 15 minutes—was taken, and when in two or three more hours' time the new steady state was attained this flow was remarkably constant, the weighings not differing by more than ·5 gm. or ·6 gm. in 300 gms. In this connexion it is interesting to note that there is an automatic tendency to maintain constancy in the flow; for any slight increase of head—due for example to vibration—temporarily causing a faster flow would introduce a higher counterbalancing pressure in the experimental tube the mercury in which on the whole would become slightly colder and heavier. Thus the equilibrium is stable.

Under the new steady state the temperatures are again noted as well as the temperature of the enclosure. All the readings were corrected so as to give the actual excesses above the temperature of the surroundings. To do this readings were taken when all the thermometers were *in situ* and at the same temperature. They then showed slight differences due to various causes, one of which was the hydrostatic pressure of the mercury.

It was found that if the ordinary stationary state was not required the steady state under flow could be more quickly obtained by starting the flow, when, in the variable state, the top thermometer reached a temperature near that desired.

If a flow from hot to cold was desired the head H and capillary C were interchanged.

(6) *Results.*

(*a*) Mean distances between thermometers by cathetometer.

t_1-t_2	6·12 cms.
t_1-t_3	11·91 cms.
t_1-t_4	17·86 cms.
t_1-t_5	23·55 cms.

(*b*) In the table below V denotes the weight of mercury outflowing in 15 minutes. The spiral steam coil was used in every case. The corrected temperatures t_1-t_5 and the enclosure temperature are given.

Magnitude of flow.	t_1.	t_2.	t_3.	t_4.	t_5.	Enclosure.
V=0 (ord. steady state)	70·9	52·0	38·2	27·1	18·5	19·5
V=114·6 gms.	66·2	45·5	31·6	22·1	15·0	18·8
V=141·9 gms.	64·8	43·5	30·2	20·9	14·2	18·5
V=323·0 gms. A	52·8	30·5	19·5	13·1	9·3	19·0
V=399·0 gms. B	47·7	26·1	16·4	11·1	8·0	19·5
V=481·25 gms. C	43·9	22·6	14·4	10·1	7·4	19·2
V=497·97 gms. D	43·1	21·5	13·5	9·6	7·1	18·0
V=543·88 gms. E	40·5	19·7	12·5	9·0	6·5	18·0
V=820·0 gms.	21·1	10·0	7·4	5·5	4·5	15·0
V=+268 gms. (flow hot to cold).	76·7	63·6	51·6	41·1	30·5	14·0

The above are shown plotted as curves (Pl. VII. fig. 2) temperatures being represented by ordinates, and distance from the first thermometer (multiplied by five) being taken as abscissæ.

(*c*) When the flow was from hot to cold the action of the spiral was harmful. Accordingly a result is given which was obtained earlier in the work with an experimental tube fitted with thermometers 7 cms. apart, the first and fifth being at the extreme top and bottom respectively of the vertical portion of the tube. This result, when plotted, shows that with a sufficient downward flow the temperature-distance curve is concave as regards the origin—a result in accordance with theory.

State of vertical column.	t_1.	t_2.	t_3.	t_4.	t_5.	Enclosure.
Ordinary steady state	75·0	53·5	35·4	22·1	10·3	19·0
Flow of 39 c.cms. in 13 mins. (hot to cold)	88·0	79·0	67·4	54·1	29·0	19·0

(7) *Calculation of Thermal Conductivity.*

The curves in section (6) *b* marked A—E fall within the range suitable for calculating the thermal conductivity of mercury. Remembering that the velocity is negative equation 5 of section (3) may be written for practical purposes:—

$$K = \tfrac{1}{2}\frac{s}{At} \times \cdot 4343 \times \left(\frac{VL}{\log_{10}\dfrac{\theta_1}{-\theta_2}}\right)$$

where V is the weight of mercury escaping in the time t, which was always 15 minutes, and θ_1, θ_2 are temperatures on the curve in question at distances $\frac{L}{2}$ on the left and right respectively of the point on the curve at the same temperature as the room.

s the specific heat of mercury $= \cdot 0333$.

A the cross-section of the tube was found $= 5\cdot 98$ sq. cms.

$t = 15 \times 60$ seconds.

The following are the values of $\dfrac{5\mathrm{LV}}{\log_{10}\dfrac{\theta_1}{-\theta_2}}$ in each case, for each curve three different values of L being taken.

Curve.	V.	5L.	θ_1.	$-\theta_2$.	Abscissa of room temp.	$\dfrac{5\mathrm{LV}}{\log_{10}\dfrac{\theta_1}{-\theta_2}}$		Mean.
A	323	2×26·3 2×39·3 2×51·3	28·3−19·0 35·2−19·0 43·4−19·0	19·0−13·4 19·0−11·4 19·0− 9·9	61·3	77120 77240 77380	=	77250
B	399	2×35 2×25 2×48	36·5−19·5 30·1−19·5 47·7−19·5	19·5−12·0 19·5−13·6 19·5−10·3	48	78610 78430 78750	=	78600
C	481·25	2×25 2×32 2×40	31·1−19·2 36·5−19·2 43·9−19·2	19·2−13·4 19·2−12·3 19·2−11·3	40	77100 77160 77770	=	77340
D	497·97	2×25 2×35 2×41	29·5−18·0 36·9−18·0 43·1−18·0	18·0−12·5 18·0−11·2 18·0−10·4	41	77750 78500 78690	=	78310
E	543·88	2×24 2×30·5 2×35·5	29·8−18·0 34·9−18·0 40·5−18·0	18·0−12·5 18·0−11·6 18·0−11·0	35·5	78750 78670 76150	=	77860

Mean = 77870

Or mean value of $\dfrac{\mathrm{LV}}{\log_{10}\dfrac{\theta_1}{-\theta_2}} = 15574.$

This gives a mean value for K = ·0209.

Each curve was drawn three times by an assistant and the one lying intermediate between the other two accepted for calculation of K. In such graphic methods a repetition by different people, or even by the same person, leads to slightly different results; but in the present experiments, where the size of the thermometer-bulbs must have an appreciable effect on the uniformity of cross-section of the tube, it would be out of place to use elaborate mathematical methods of interpolation which will most probably be necessary in future work. A general consistency for the value of K is obtained using different velocities, and this value is of the right order.

(8) *Conclusion.*

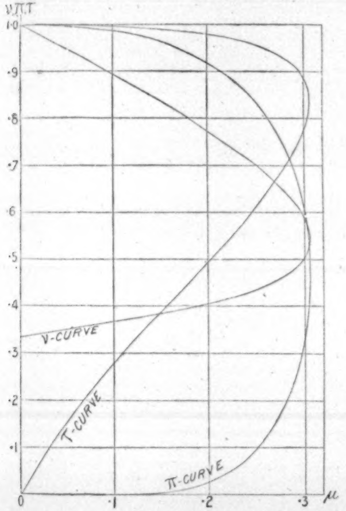
The curves obtained agree closely with those which can be deduced from theory, and promise to lead to a new method of determining thermal diffusivity directly. This method has the advantage of requiring only temperature measurements, quantity of heat not entering directly but being inferred from a knowledge of specific heat. In improving the method the first thing will be the substitution of thermo-junctions—at much more frequent intervals—for thermometers. The cross-section of the tube need hardly then be affected, and many more points on the curve could be obtained, thus diminishing very much errors in graphic interpolation.

The author is, however, presenting his research at this stage, embracing the theory and the work which can be done with thermometers, as he is anxious to perform experiments under the best possible conditions, and hopes to receive advice and criticism before proceeding with more delicate work.

The author wishes to express his thanks to the Principal and Council of Birkbeck College for facilities for carrying out the work, and more especially to Dr. A. Griffiths who has watched the work with interest from the beginning and has given valuable help and encouragement throughout.

FIG. 1.

Corresponding Values of ν, π, τ in terms of μ for Saturation with van der Waals' characteristic.



$$\log_e \frac{3\nu+1+\mu}{(3\nu-1)(3\nu-1-\mu)} = \frac{3\nu(9\nu-1+\mu)(3-3\nu+\mu)}{(3\nu-1)(3\nu+1-\mu)(3\nu+1+\mu)}$$

$$\pi = \frac{(3\nu-1-\mu)(3\nu-1+\mu)}{4\nu^3}$$

$$\tau = \frac{(3\nu-1)(3\nu+1-\mu)(3\nu+1+\mu)}{32\nu^3}.$$

FIG. 2.

Corresponding Values of π, ν, ν' in terms of τ for Saturation with van der Waals' characteristic.

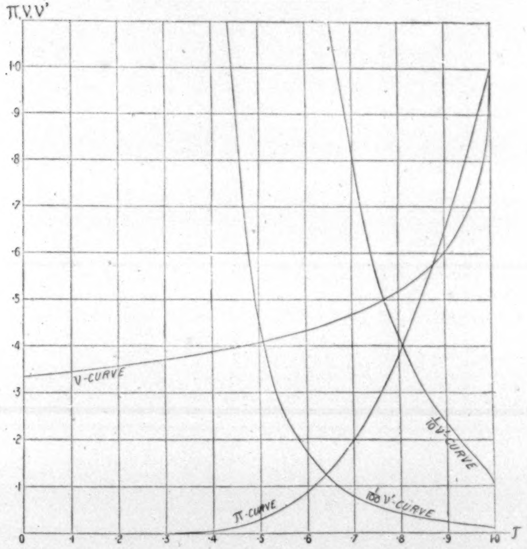


FIG. 3.

Latent and Specific Heats for Saturation in terms of τ with van der Waals' characteristic.

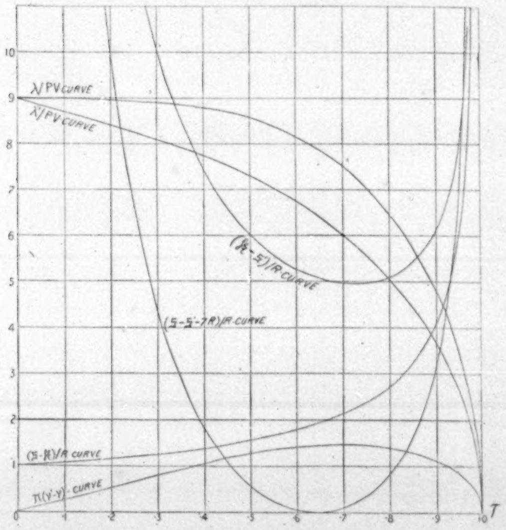


FIG. 4.

Latent and Specific Heats, &c. for Saturation in terms of τ. Clausius' characteristic.

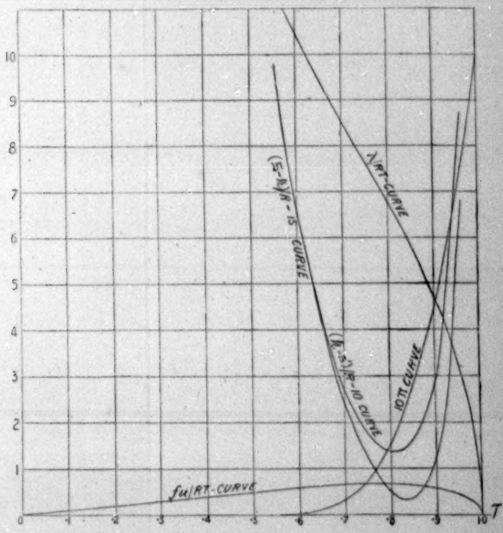
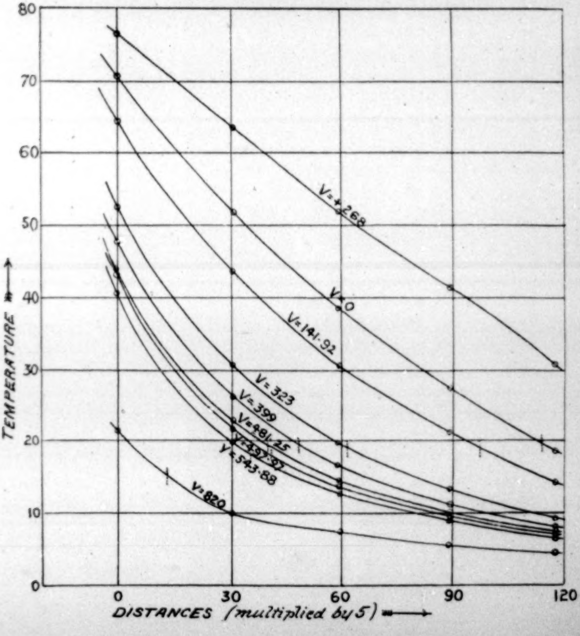


FIG. 1.

FIG. 2.


80
70
60
50
40
30
20
10
0
TEMPERATURE →
0
30
60
90
120
DISTANCES (multiplied by 5) →
V=+268
V=0
V=141·92
V=323
V=399
V=481·25
V=497·97
V=543·88
V=820

XX. *On Coherers.* *By* W. H. Eccles, *D.Sc., A.R.C.S.*[*]

It is well known that an imperfect contact between almost any pair of good electrical conductors suffers a change of resistance when electrical oscillations of sufficient intensity occur in the circuit containing the contact. Prof. Branly, in 1896, concluded after much investigation that the effects observed were due to a modification of the electrical conductivity of the film of dielectric that happened to separate the conductors. For example, a mass of metal filings melted down in resin showed appreciable alteration of resistance when a spark discharge occurred near it; and a pair of pieces of metal in very light contact, yet separated by their surface films of condensed gas, showed the phenomenon excellently. Sir Oliver Lodge at that date held the view that the alteration of resistance, usually a diminution, was due to the piercing of the dielectric and a consequent cohesion of the metals. He gave the name "coherer" to instruments exhibiting or utilising the phenomenon, and this name has been universally adopted and the suggested explanation widely accepted. It has often been pointed out that Lodge's explanation is inapplicable to known cases where the resistance of the coherer is increased by the passage of electrical oscillations. In the present paper it will be shown that there is no need to adopt the "cohesion" hypothesis, even in the case of typical coherers where the resistance is lowered by the action of electrical oscillations.

Ten years ago the known ways of investigating a coherer required the device to submit itself to powerful electric discharges. We find investigators of filings coherers watching the sparks between the filings and boasting of fusing the filings into chains of quite considerable tenacity. But the engineers who were at that date nursing the infant wireless telegraphy were, on the whole, more gentle with the coherer than were the men of science. At most, the engineers required that the current through the coherer when it "cohered" should work a delicate relay. A few years later they merely asked that the coherence current should move the diaphragm of a telephone receiver. The oscillatory voltages that need be applied to a coherer in order to produce a resistance change perceptible by an ear at the telephone are very small indeed; and in these cases a good coherer spontaneously and very perfectly recovers its original resistance when the oscillations cease. The "cohesion" explanation appears improbable here, though in the cases familiar ten years ago, where the alteration of resistance is permanent, that explanation seemed plausible.

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[*] Read March 11, 1910.

Within the past six years the problem of the coherer has often been attacked, and each observer has in turn found all the theories of the action of coherers insufficient to explain his experiments, and has added some suggestion of his own. For example, A. Blanc[*] in 1904 concluded that neither oxide nor condensed gases played a part in certain coherers, and believed that particles of the two surfaces diffuse at a rate depending on the current, temperature, and pressure. On the other hand, Robinson[†] decided that in single point coherers the resistance is that of an elastic film of oxide, and that permanent coherence implied injury or thickening of the oxide; while Huth[‡] attributes coherence to the ionization of the film rather than to the thickening of the oxide. From the point of view of the results of some of the experiments to be described in this paper, the observations of A. H. Taylor[§] are of special interest. He found that the current through a single point steel coherer shows three stages as the gap is diminished. In the first stage there is a feeble leakage current due, he thinks, to vapour conductivity; in the second, conduction due to metallic ions (in this stage a small rise of E.M.F. produces a great increase of current); in the third stage ordinary conduction occurs. In a paper entitled "Recherches sur les Contacts Imparfaits" (*Journ. de Physique*, ser. 4, iii. 1904, p. 350), A. Fisch describes his experiments on the voltage-current curves for a coherer consisting of two spherical surfaces of steel whose separation was capable of fine adjustment.

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[*] Science Abstracts, vol. viii. 1923, 2248.

[†] Science Abs. vol. vii. No. 1135; *Ann. d. Phys.* xi. 4, p. 754 (1903).

[‡] Science Abs. vol. vii. No. 1136; *Phys. Zeit.* iv. p. 594 (1903).

[§] Science Abs. vol. vii. No. 157; Phys. Review, xvi. p. 199 (1903).

He found, from a study of numerous curves, that the average curve exhibits three stages:—(1) As the current is raised the potential difference across the contact increases more and more slowly to a critical value constituting stage (2), where, though the current increases, the potential difference remains constant. Then (3) increasing current is accompanied by continually rising potential difference. This curve of Fisch's is precisely the curve II. of fig. 8, below. Besides these investigators there are others who have advocated that the electrostatic attraction between the conducting surfaces may account for the phenomena.

Very few, if any, experiments with electrical oscillations of very minute measured voltages have so far been published, yet only minute voltages nowadays arise in practice. The purpose of the present investigations is to examine the action of a certain kind of coherer under "weak signals," especially in respect of their energy relations.

Method of Experiment.

The self-restoring coherer to be examined was placed in a circuit comprising the necessary source of steady electromotive force and a telephone receiver. At the same time it was also connected in series with a condenser, to form, as in practical wireless telegraphy, a shunt across the condenser of an oscillatory circuit (fig. 1). This oscillatory circuit was made a secondary circuit in relation to a tuned primary oscillatory circuit. The coupling was always extremely loose. Trains of damped oscillations, of maximum amplitude of the order of a few hundredths of an ampere could be excited in this primary; and hence feeble electrical oscillations of calculable magnitude could be excited in the secondary. The amplitude of the oscillatory electromotive force at the coherer contact, and the energy delivered to the coherer, can be computed from the amplitude of the primary oscillations; but, besides other considerations, the lack of facilities for measuring the small mutual inductance between primary and secondary made another method (described below) of determining the energy seem preferable.

The trains of damped oscillations in the primary were produced by making and breaking a small current running through the primary inductance. In consequence, the coherer yielded a sound in the telephone connected in series. As a general rule, the telephone sound is greater the greater the amplitude of the oscillations in the primary, because the response of the coherer is greater. To measure the response of the coherer, the intensity of the sound was balanced against another sound which was produced in the same telephone by the interruption of an independent source of direct current, which could be switched into the circuit of telephone and interruptor in the place of the coherer. In this way the behaviour of the detector could be quantitatively recorded when the steady current through it, or the oscillatory energy delivered to it, underwent desired variations. The method has enabled me to attain some degree of consistency in measuring the effects of oscillatory voltages as low as $\frac{1}{40}$ volt, and energy transformations of the order of a few thousandths of an erg.

Only one type of coherer has been examined closely. This is a mercury and oxidized iron coherer—a kind I have used for many years—and chosen for these experiments after careful comparisons with a number of similar coherers of more modern vogue. As is well known, almost any conductor that can be coated with a closely adherent, tenacious, and very thin film of solid or liquid of not too high resistivity makes a good coherer: for example, oxidized or sulphided copper against any solid conductor; oxidized iron, steel, nickel, tantalum, or any similar metal, against mercury or other conductor are all good *. I have tried also iron-mercury coherers made by the action of organic acids and other reagents on iron. For example, even the graphite (?) films left on a steel knife that has been unwisely used as a fruit-knife make coherers. Amongst all these and others I have found that the best for my present purposes is the coherer made by dipping the end of a carefully oxidized fine iron wire or steel needle into a pool of clean mercury. It is important that the layer of oxide on the wire should be of uniform very small thickness. Seven or eight years ago I found that this can be accomplished very well by heating

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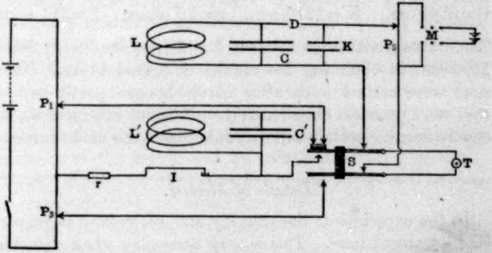
* See Branly, *Comptes Rendus*, cxxxiv. pp. 347, 1197 (1902).

the wire in a glass tube carrying steam from a boiling flask. The wire should be watched till the film of oxide yields one of those interference colours familiar in the tempering of steel. For instance, a blue film makes a splendid coherer, which works well for a month or two in the open air. Most of the experiments discussed below were carried out on coherers made in this way.

Details of the Apparatus.

Fig. 1 is a diagrammatic plan of the apparatus. The primary circuit consists of the coil L′ and condenser C′; the

Fig. 1.



secondary of L and C. The detector D and condenser K are arranged shuntwise to the condenser C. C and C′ are variable condensers of the Marconi pattern, K is an air condenser, L and L′ are coils wound on waxed cardboard tubes. The six-contact key S is designed to throw the telephone T quickly from the detector circuit to the direct current circuit. In the position of S shown in the diagram the telephone is in series with the detector and part of the potential-divider P_2, and at the same time the primary inductance L′ is connected through portion of the potential-divider P_1 and the interrupter I. When the key S is depressed the telephone is connected directly through the interruptor I and the potential-divider P_3, and the secondary circuits are disconnected. The mutual inductance between primary and secondary can be varied by running L′ along a straight slipway to or from the coil L; but usually the coils were held so that their mutual inductance was a few hundreds of centimetres. The resistance r is an important detail. Its purpose is to prevent the formation of oscillations in L′ at the *make* of the primary current; calculation (as well as experiment) shows that if r lies within certain limits oscillations occur in L′ at the *break* only. The amplitude of the oscillations is governed by aid of the potential-divider P_1. The frequency of the interruptor I was chosen of the low value 22 per second; thus resonance with the telephone diaphragm was avoided, and the balancing of the sound produced by the detector with that produced by P_3 facilitated. The electrical dimensions of the apparatus were as follows:—

$L' = 33{,}700$ cm., $C' = 5020$ cm., $L = 42{,}700$ cm.,
$C = 3940$ cm., $K = 1210$ cm., $r = 20$ ohms, $T = 311$ ohms.

The mutual inductance between L and L′ ranged from 200 to 2500 cm. in obtaining the results described below. These sizes were settled upon after many changes, partly because they were practical sizes (wave-length about 820 metres), but chiefly because certain later calculations were made easier.

Details of Method.

In the experiments the primary and secondary were used slightly out of tune. This is very necessary when detectors whose resistance varies are under study; for the variations of a coherer's resistance are quite wide enough to carry the secondary circuit erratically from one side to the other of perfect resonance if the circuits are initially very near that condition, with the result that inconsistent measurements would be inevitable. It is, of course, as easy to deduce formulæ for the mistuned as for the tuned circuits. The expression for x, the current through the detector at break of primary current, is of the form

$$x = le^{-at} + he^{-mt}\cos(pt+q) + ke^{-bt}\cos(ct+d),$$

where m and p belong to the primary circuit and b and c to the secondary circuit, the coupling being very feeble. The coefficients l, h, k are complicated functions of the electrical dimensions of the circuits. Hence the energy delivered to the detector at a break appears as six terms, of which, with the dimensions adopted, only two are of numerical importance, namely,

$$h^2/4m, \quad 2lh/m.$$

The other terms are less than 1 per cent. of the larger of these. Both h and l involve m, the damping factor of the primary, but not prominently. This fact enabled m to be determined easily and with sufficient accuracy. A calibrated thermogalvanometer with a heater and series resistance totalling 1000 ohms was put in the usual place of the detector; the mutual inductance between the circuits was increased (but the coupling was still very small), the speed of the interruptor was raised, and heavier primary currents were used than was customary when a detector was in position. Thus a considerable deflexion of the thermogalvanometer was obtained and recorded. Then a short loop of very thin copper wire of known high frequency resistance was put in series with L′, and the observation repeated. Since the energy received by the thermogalvanometer is proportional inversely to the resistance of the primary on each occasion, the high-frequency resistance of the coil L′ can be calculated from the observations. The mean of several determinations gave 0·78 ohm. The steady-current resistance was 0·22 ohm.

The power delivered to the detector in the standard conditions of the circuits and primary currents can be computed as just indicated. For accurate results it is necessary to know M, the mutual inductance between primary and secondary, and m, the primary damping factor, very accurately. As neither of these could be determined with the precision desirable, the thermogalvanometer was invoked to determine the power passing from primary to secondary. In fact, the experiment to determine m, just described, may be taken as determining the factor of transference of power. Similar observations to the above were therefore made with primary currents of different values and with rather high speeds of interruption (nearly 200 per second), and the fact was established, obvious theoretically, that the power passed to the thermogalvanometer was proportional to the square of the primary current and to the frequency of interruption. Finally, by observation of the thermogalvanometer deflexion when a primary current of $\frac{1}{10}$ ampere was interrupted

168 times a second, the result was established that with the primary and secondary in their standard relation a quantity of energy amounting to very approximately $4\cdot82 \times 10^{-11}$ joule, or about 0·0005 erg, is passed to a detector of 1000-ohms resistance at each break of a primary current of 0·01 ampere. This corresponds to a power-delivery of 0·01 erg per second when the interrupter is running at the standard frequency. Corresponding figures were obtained for the cases where the detector has different resistances between 100 ohms and 2000 ohms.

In the actual experiments on detectors two things were chosen for variation, namely, the electromotive force applied to the detector and the oscillatory energy passed from the primary to the detector. The effect observed was the loudness of the sound in the telephone. The curves of fig. 2 (p. 297) have as abscissæ the varying external E.M.F. applied to the coherer, and as ordinates the consequent effect in the telephone. Each curve in the diagram is obtained by supplying to the coherer the constant power marked on the curve. The curves in fig. 3 (p. 298) have as abscissæ the power supplied to the coherer when its applied E.M.F. is unvarying ; the ordinates are the effect in the telephone. The telephone effect is plotted as estimated power given by the coherer to the telephone circuit. The numerical connexion between the power passed into the telephone circuit and the measured intensity of the sound in the telephone was determined as follows. The thermo-galvanometer was put in the telephone circuit in series with the telephone, and a large primary current was interrupted at the normal rate. The intensity of the loud sound in the telephone was measured by the potential divider, and at the same time a reading of the galvanometer deflexion was taken. A resistance was then put in series with the telephone and the galvanometer, and the measurements repeated. These readings give sufficient data to determine the amount of energy being dissipated in the telephone when the intensity of a sound is equal to that due to any assigned length of the potential divider P_3. The measurements thus made showed, besides, that the power delivered by the detector to the thermo-galvanometer and telephone was practically proportional to the square of the length of P_3 required to produce equal sounds in the telephone. It is rather a big extrapolation to

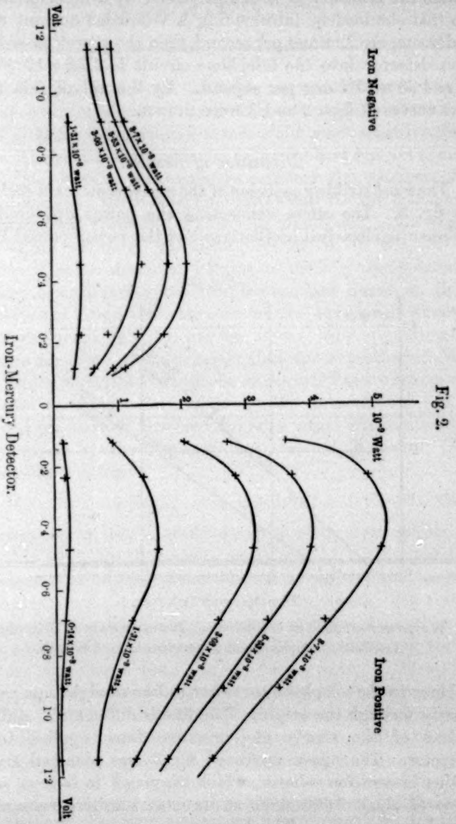


Fig. 2.

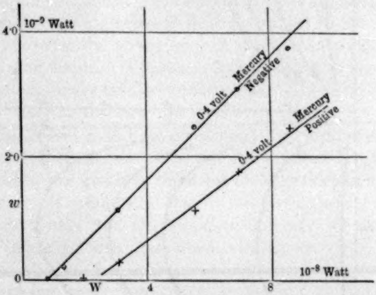
Iron-Mercury Detector.

extend these results obtained with currents large enough to affect the thermogalvanometer, to currents that give only faint sounds in the telephone; but if we do this, we find that when the intensity of sound produced by a detector is equal to that obtained by interrupting a telephone current of one microampere 22 times per second, then the power passed from the detector into the telephone circuit is $4{\cdot}38 \times 10^{-12}$ watt, *i. e.* $4{\cdot}38 \times 10^{-5}$ erg per second. By the aid of this factor, the curves of figs. 2 and 3 were drawn.

Discussion of Results.

The most striking outcome of the measurements is embodied in fig. 3. The curve connecting the power delivered to a coherer as electrical oscillations and the power passed by the coherer to the telephone turns out to be a straight line passing nearly through the origin. The line is different for different values of the steady electromotive force applied to the detector. The upper curve of fig. 3 was obtained from a rather insensitive coherer, which happened to have a resistance of about 1000 ohms at its most sensitive voltage, the iron being positive. The lower curve is one selected under similar considerations with the iron negative. The approxi-

Fig. 3.



Iron-Mercury Detector.

W = power supplied as oscillations. Resistance about 1000 ohms.
Oscillation amplitude at coherer $= 442 \times \sqrt{W}$ volt.

mate equation of the former is

$$w = 0{\cdot}024(W - 0{\cdot}9 \times 10^{-8}),$$

of the latter is

$$w = 0{\cdot}019(W - 2{\cdot}2 \times 10^{-8}),$$

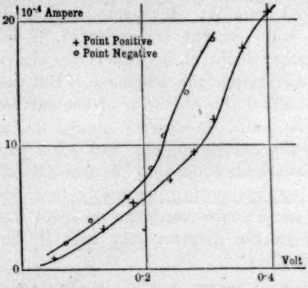
the unit being a watt. All the other curves that have been obtained with coherers of various construction, resistance, applied E.M.F., are very like these. The more sensitive the coherer the steeper the line, is the only rule that has yet come to light (and in some iron-mercury coherers the gradient of the line is 0·06); there is no rule yet proved for the constant term subtracted from W. It may be mentioned again that the absolute value of the multiplier of W in the above equations is rather doubtful, because of the big extrapolation necessary in calibrating the telephone. But doubt on this point does not affect the character of the curves; it affects only their gradient. As written above, the approximate equations for the curves suggest that for a particular detector under invariable conditions there is a fixed wastage of oscillation energy amounting constantly to about one-tenth of an erg per second, however large or small the oscillation energy given to the detector may be. If, however, the equations be written

$$w + 0{\cdot}022 \times 10^{-8} = 0{\cdot}024W \quad \text{and} \quad w + 0{\cdot}042 \times 10^{-8} = 0{\cdot}019W,$$

the inference is that a small quantity of energy, which is invariable while the detector is undisturbed, is delivered by the detector to the telephone circuit in a form that never makes any proportion of itself manifest as sound. Whatever the cause of this small energy wastage may be, the curves of fig. 3 show that the coherer must be put amongst those detectors that have been called "integrating" detectors. Thus the coherer is not, as has usually been supposed, a "voltage-operated" detector. As detectors go, it has a very good efficiency (*i. e.* ratio of energy conversion), but evidently its efficiency falls rapidly as signals get weak.

The curves of fig. 2 show how the sound heard in the telephone varies with alteration of the potential difference applied to the coherer and telephone in series. Each curve is obtained by submitting the coherer to the fixed excitation whose value in watts is marked on that curve. As the excitation diminishes the maxima accompanying variation of applied voltage become less marked. The resistance of the coherer was kept as constant as might be during these experiments, but wide variations of resistance are inevitable. The asymmetry of the coherer under positive and negative voltages shown in these curves obtained by the action of electrical oscillations on the instrument, is no doubt connected with the similar phenomenon mapped in fig. 4, where typical steady

Fig. 4.



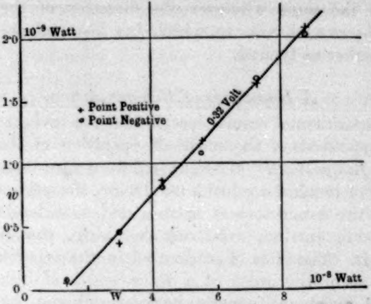
Iron-Mercury Detector.—Steady-Current Curves.

current curves, plus and minus, are drawn. In these curves the abscissæ are the potential differences applied at the terminals of the coherer. In both cases the lack of symmetry is due to the use in the coherer of two metals so different as iron and mercury. In order to examine whether this asymmetry was responsible for the failure of the curves of fig. 3 to pass through the origin, a new series of experiments on a different form of iron-oxide coherer was undertaken.

Additional Experiments.

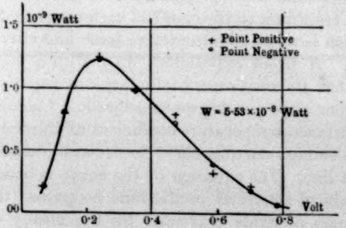
A coherer was made by heating a polished iron plate in air till rings of colour round the small heated area indicated that the plate was covered with a thin film of oxide. The end of a clean piece of iron wire was so held as to press on the plate at any desired place with a constant force. The plate was made one terminal of the coherer and the wire the other terminal, and this coherer was examined in precisely the same way as the others. A typical series of measurements is set forth in figs. 5 and 6. The curve connecting the energy given to the coherer and the energy passed to the telephone is again a straight line passing some distance from the origin.

Fig. 5.



Iron Point on Oxidized Iron Plate (light blue film).
Applied voltage, 0·32.

Fig. 6.



Iron Point on Oxidized Iron Plate.

The equation of the curve is

$$w = 0{\cdot}028(W - 1{\cdot}4 \times 10^{-8}).$$

The curve is the same whether the current passes from point to plate or in the opposite direction, which clearly indicates that the asymmetry of the other curves was due solely to the use of two metals. The curves connecting applied voltage and steady current for these iron point iron plate coherers are also the same whatever the direction of the current. Fig. 7 shows such a curve, which has been selected from a large number as typical.

A Hypothesis of Coherer Action.

The experimental results described above lead me to offer a simple hypothesis of the mode of operation of the class of coherers in question. When an oxidized wire is dipped into mercury or touched against a conductor, the oxide dielectric between the conductors is so thin and so extended that a considerable current, relatively speaking, can be passed through it. The films of oxide used in the experiments had a thickness of the order of a wave-length of light and an electrical resistance usually between 50 and 2000 ohms. When a current is passed through the film, heat is generated in the film and raises its temperature. The rise of temperature may be considerable if the mass heated is small, but is limited by the escape of heat to the surrounding metal. If the rise of temperature does not affect the resistance of the heated material appreciably, the relation between the applied electromotive force and the consequent current will be a linear one; but if the resistance of the material varies with its temperature the relation between electromotive force and current will not be linear. A direct experiment on a film of oxide of iron showed that its resistance coefficient was negative and more than 0·1 per cent. per degree centigrade. Calculation shows that a resistance-temperature coefficient of this magnitude can cause the voltage-current curve to deviate considerably from a straight line. The equation of the curve is deduced below.

If a train of electrical oscillations be passed through the minute mass of oxide traversed by the steady current, its energy appears as heat in the oxide. The resistance falls in consequence, and the equilibrium of the direct current in its circuit is disturbed. Since the rise of current enhances the heating of the oxide, the subsidence of the current to equilibrium is somewhat prolonged when, as in the case of iron-oxide, the resistance-temperature coefficient is negative.

Hence the effect on a telephone diaphragm, whose natural period of vibration must be regarded as great in comparison with the duration of a train of oscillations, may be very large. It remains to be seen whether the disturbance of the direct current, caused by a small variation of resistance, can account for the phenomena disclosed in the course of the above-described experiments.

Let W_1 be the fraction of the energy of a single train of oscillations given to the variable resistance ρ of the detector; let $\rho=\rho_0(1-\alpha\theta)$ where ρ_0 is the value of ρ at the temperature of the surroundings. Then $kW_1=(\delta\theta)=-(\delta\rho)/\rho_0\alpha$, where k is a constant involving Joule's equivalent and the specific heat and the mass of the oxide. This initial disturbance $(\delta\rho)$ of resistance is accomplished in less than 1/10000 of a second and gives rise to the perturbations of current to be traced immediately.

Let the direct current circuit of the detector comprise a total invariable resistance r (which includes the resistance of the telephone), the variable resistance ρ of the detector, and the inductance L of the telephone. Let c be the current at any moment and ϵ the applied electromotive force. Assume that the rate of loss of heat from the warmed mass of oxide to its surroundings is $m\theta$, where, as implied above, θ is the temperature of the oxide above the temperature of the surroundings, taken as zero. Then at any time t

$$\frac{d\theta}{dt}=k\rho c^2-m\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$L\frac{dc}{dt}+(r+\rho)c=\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since $\rho=\rho_0(1-\alpha\theta)$, (1) becomes

$$\frac{d\rho}{dt}+k\rho_0\alpha\,.\,\rho c^2+m\rho=m\rho_0. \quad . \quad . \quad . \quad . \quad (1')$$

The simultaneous equations (1′) and (2) contain the complete history of the fluctuations of c and ρ from any assigned equilibrium condition. We need only investigate here *small* fluctuations. First, however, let us obtain the equilibrium value of c, say c_1, corresponding to a given value of the applied electromotive force ϵ. Put

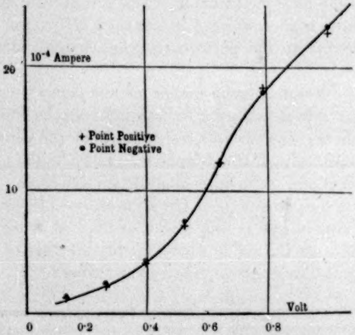
$dc/dt=0$ and $d\rho/dt=0$ in (1′) and (2) and eliminate ρ. We get

$$\epsilon=\left(\frac{m\rho_0}{m+k\alpha\rho_0c^2}+r\right)c. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

This equation agrees perfectly with the results of numerous experiments on the iron-oxide-iron coherer. For example, the experimental curve of fig. 7 has the parameters

$$r=390 \text{ ohms}, \ \rho_0=970 \text{ ohms}, \ \frac{k\alpha}{m}=3{\cdot}7\times10^6.$$

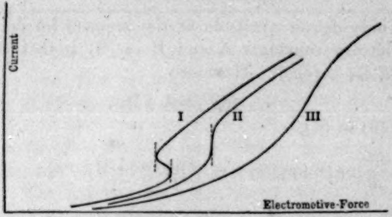
Fig. 7.



Iron point on oxidized iron plate. Steady current curve.

In general it would appear that the curves drawn from experiments on detectors should fall into the two classes indicated in fig. 8 by curves I., III., separated by the case II. In class I. the curve has two vertical tangents; the condition for this is $\rho_0>8r$. If when the coherer is in use the values of ϵ and c pass those corresponding to the lower bend of the curve, the original values of c and ϵ can only be regained by interrupting the current by external agency: the coherer is not "self-decohering." In class III. $\rho_0<8r$: there is no vertical tangent, and the coherer is self-restoring. In case II. $\rho_0 = 8r$. All these cases can be realized, with patience.

Fig. 8.



Ideal Curves suggested by theory.

To return to the problem of the small fluctuations of the current from its equilibrium value when the arbitrary small variation ($\delta\rho_1$) is imposed on the system by the arrival of a train of oscillations at the time $t=0$. Equations (1′) and (2) become

$$L\frac{d\,\delta c_1}{dt} + (r+\rho_1)\delta c_1 = -c_1\delta\rho_1 \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{d\delta\rho_1}{dt} + n\delta\rho_1 = -2(n-m)\frac{\rho_1}{c_1}\,\delta c_1. \quad . \quad . \quad (5)$$

Here $$n \equiv k\rho_0\alpha c_1{}^2 + m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and ρ_1, c_1 are corresponding equilibrium values of ρ and c.

Equations (4) and (5) give

$$L\frac{d^2\delta\rho_1}{dt^2} + (Ln+r+\rho_1)\frac{d\delta\rho_1}{dt} + \{n(r+\rho_1) - 2(n-m)\rho_1\}\delta\rho_1 = 0,$$
$$. \quad . \quad . \quad (7)$$

which can be written in the form

$$[D^2 + aD + b]\delta\rho_1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Here a is essentially positive. The quantity b has the same sign as $d\epsilon/dc$, as may be seen by differentiating (3) and comparing (7) and (8). Since my experiments have been for the most part confined to self-restoring coherers, b has been in practice always positive. Thus (8) yields

$$\delta\rho_1 = A\,e^{-m_1t} + B\,e^{-m_2t}, \quad . \quad . \quad . \quad . \quad (9)$$

where $$m_{1,2} = \frac{a}{2}\left(1 \pm \sqrt{1 - \frac{4b}{a^2}}\right).$$

It appears later that a^2 is always much greater than $4b$, so that, approximately,

$$m_{1,2}=b/a \quad \text{or} \quad a-b/a.$$

The only datum available at the moment for determining the arbitrary constants A and B in (9) is that the initial value of $\delta\rho_1$ is $(\delta\rho_1)$. Therefore

$$(\delta\rho_1)=\mathrm{A}+\mathrm{B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Using (9) in (4),

$$\mathrm{L}\frac{d\delta c_1}{dt}+(r+\rho_1)\delta c_1=-c_1(\mathrm{A}e^{-m_1t}+\mathrm{B}e^{-m_2t})\,;$$

whence

$$\delta c_1=-\frac{c_1}{\mathrm{L}}\left[\mathrm{A}\frac{e^{-m_1t}-e^{-\frac{r+\rho_1}{\mathrm{L}}t}}{\frac{r+\rho_1}{\mathrm{L}}-m_1}+\mathrm{B}\frac{e^{-m_2t}-e^{-\frac{r+\rho_1}{\mathrm{L}}t}}{\frac{r+\rho_1}{\mathrm{L}}-m_2}\right], \quad . \quad . \quad . \quad (11)$$

which vanishes at $t=0$.

Let the work done in the telephone by the fluctuation of the current through it be w_1, and let the effective resistance of the telephone be P. Thus the rate at which the telephone current is working at any moment is $\mathrm{P}c_1^2$, and, therefore,

$$w_1=\delta\mathrm{P}\int_0^\infty c_1^2dt=2\mathrm{P}c_1\int_0^\infty \delta c_1dt.$$

Equation (11) now gives

$$\begin{aligned} w_1&=-\frac{2\mathrm{P}}{r+\rho_1}c_1^2\left\{\frac{\mathrm{A}}{m_1}+\frac{\mathrm{B}}{m_2}\right\} \\ &=-\frac{2\mathrm{P}}{r+\rho_1}\cdot\frac{c_1^2}{m_2}\left\{(\delta\rho_1)+\mathrm{A}\left(\frac{m_2}{m_1}-1\right)\right\}, \quad . \quad (12) \end{aligned}$$

by using the condition (10).

Now, as stated before, $(\delta\rho_1)=-k\rho_0\alpha\mathrm{W}_1$, where W_1 is that portion of the energy of a single train of oscillations usefully converted to heat in the detector. Therefore (12) becomes

$$w_1=\frac{2\mathrm{P}k\rho_0\alpha}{r+\rho_1}\cdot\frac{c_1^2}{m_2}\left\{\mathrm{W}_1-\frac{\mathrm{A}}{k\rho_0\alpha}\left(\frac{m_2}{m_1}-1\right)\right\}. \quad . \quad (13)$$

The experiments have shown that the term involving the arbitrary constant A is always positive. Hence, if A be positive, m_2 must be greater than m_1; moreover, (10) shows that B must be negative when, as was always the case, $(\delta\rho_1)$ is negative. These considerations applied to equation (9) imply that $\delta\rho_1$ runs through a series of diminishing negative values, passes through the value zero, and becomes positive. This is not possible. On the other hand, if A be negative, m_2 must be less than m_1, and B may be either positive or negative. If B be positive, (10) shows it must be numerically less than A, and then the condition $m_2 < m_1$ implies that $\delta\rho_1$ in equation (9) changes sign once during its history. Rejecting this case we are left with A and B both negative and $m_2 < m_1$. The quantity m_2 must, therefore, be given the value b/a and m_1 must equal $a - b/a$. Put $A = -f^2$, and (13) becomes

$$w_1 = \frac{2Pk\rho_0 a}{r+\rho_1} \frac{ac_1^2}{b} \left\{ W_1 - \frac{f^2}{k\rho_0 a} 1 - \left(\frac{b}{a^2 - b} \right) \right\}. \quad . \quad (14)$$

This equation regarded as an equation between the energy W_1 spent in the detector and the energy w_1 delivered by the detector to the telephone, is precisely of the form required by the experimental curves of figs. 4 and 5. Those curves may be regarded, in fact, as determining the constant of integration f^2. The methods of measurement are not yet sufficiently refined, however, to make it worth while to examine this intercept term in detail. The coefficient of W_1, on the other hand, may be profitably enquired into.

In this coefficient P, k, ρ_0, a, and r (which includes P, of course) are to be taken as constants, while ρ_1, a, b have values dependent on the square of c_1, the direct steady current passing round the detector circuit. By aid of equations (6), (7), and (8) the part of the coefficient which varies with c_1 may be expressed in terms of the quantity n defined by (6). This quantity n, it should be noticed, is itself a function of the square of c_1, and therefore the following discussion applies equally to positive and negative steady currents through the detector. It will be found that

$$Lan = Ln^2 + m\rho_0 + rn$$

and

$$Lb = n^2 r - mn\rho_0 + 2m^2\rho_0, \quad . \quad . \quad . \quad (15)$$

when, as implied by (3), we remember that

$$m\rho_0 = n\rho_1.$$

Now if (3) be differentiated (15) will be seen to become

$$\mathrm{L}b = n\frac{d\epsilon}{dc}.$$

Hence the coefficient of W_1 in equation (14)

$$\propto (n-m)\left(\frac{\mathrm{L}n}{rn+m\rho_0}+\frac{1}{n}\right)\frac{dc}{d\epsilon}. \quad . \quad . \quad . \quad (16)$$

It is convenient to think of c_1, the impressed current through the detector, as the independent variable. It is, of course, readily connected with ϵ, the impressed voltage by equation (3) or by fig. 7.

Then as c_1 is varied the factor $n-m$ has as graph a parabola with its vertex at the origin and with the tangent there horizontal. This parabola rises only slightly within those values of the current ever passed through detectors. The second factor has as graph, when values of c_1 are again taken as abscissæ, a slowly falling curve. As a rule the graph of the product of these two factors appears to be a slowly rising curve. Now the third factor $dc/d\epsilon$ is the gradient of the ϵ, c, curves shown in figs. 7, 8. Hence, for a self-restoring coherer we expect that if we draw curves connecting w and ϵ—that is the watts in the telephone, and the voltage applied steadily to the detector—we expect to find w rising to a maximum at a value of ϵ rather beyond the steepest part of the steady-current ϵ, c, curve. This deduction is amply borne out by the experimental curves of figs 2 and 6. I am not able at the present moment to enter into an exact numerical examination of the whole matter; but such rough values as have as yet been obtained leave little room for doubt that the quantitative adequacy of the above reasoning is as perfect as its qualitative sufficiency. It may be remarked, in closing, that the term subtracted from W_1 in the bracket of equation (14) has its maximum at about the same value of c_1 as gives the maximum gradient in the ϵ, c, curve of the detector. This deduction also has to some extent been experimentally confirmed, but the lability of coherers is so remarkable and the difficulty of carrying a coherer through a variety of experiments without altering its internal state is so great, that full confirmation will only be attained by selection of the best from a very large number of observations.

I have to thank the Royal Society's Government Grant Committee for a grant of money to defray the cost of construction of the apparatus used in the above experiments.

Summary.

A method of investigating detectors is developed with special reference to the relations between the energy given to the detector in the form of electrical vibrations and the energy delivered by the detector, as direct current, to the circuit of the indicating instrument. The stream of energy supplied to the detector was always of the same order as that usual in telegraphy. The detector under examination was placed in a circuit containing suitable inductance and capacity, which was secondary to a primary circuit. The primary could be set into electrical vibration by breaking a known current in it. The coupling was very small, so that when a current of a few milliamperes was broken in the primary, the energy delivered to the detector was of the order a thousandth of an erg, and the electromotive force at the coherer terminals was of the order a tenth of a volt. The response of the detector was measured by comparing the sound in its telephone with the sound produced in the same telephone by interrupting a measurable direct current. A special switch key enabled the comparison to be made quickly. The power delivered to the detector and to the telephone was determined by extrapolation from measurements on stronger currents with the thermogalvanometer.

The results of experiments on coherers made of oxidized iron wire dipping into mercury, and on coherers made of a clean iron point touching an oxidized iron plate, are exhibited as curves connecting: (1) the steadily applied E.M.F. and consequent current through the coherer; (2) the steadily

applied E.M.F. and the power given to the telephone, for various rates of delivery of vibration energy to the detector; (3) the power delivered to the detector and the power passed to the telephone, the E.M.F. applied to the coherer being constant. Curves (1) show that in a self-restoring coherer the current increases more and more rapidly as the E.M.F. is raised, till, in general, a point of inflexion is reached, and then the current increases more slowly. Curves (2) show the rise and fall of sensitiveness to oscillations as the applied E.M.F. is increased. Curves (3) show that if W represents the power in watts delivered to the coherer, and w the power passed to the telephone circuit, then $w=m(W-a)$ where m and a have values settled by the magnitude of the current through the detector. The quantity m for a good low resistance iron-mercury coherer has been found to be as high as 0·06; while a is usually near $1{\cdot}0 \times 10^{-8}$ watt. These curves show that these coherers are not "voltage-operated" detectors but "integrating" detectors.

The author puts forward the hypothesis that the properties of an oxide coherer may arise solely from the temperature variations caused in the minute mass of oxide at the contact by the electrical oscillations and by the applied E.M.F. He examines the hypothesis mathematically, and shows that most of the phenomena recorded in the curves (1), (2), (3) above, can in this way be accounted for as perfectly as the present state of the measurements permits.

ABSTRACT.

A summary of the paper is given on pp. 309, 310 above.

DISCUSSION.

Mr. W. **DUDDELL** expressed his interest in the paper, and remarked that it was important to know how much energy was needed to work a detector in order to calculate the amount which it was necessary to radiate. He asked the Author if the action of the coherer he had described might not depend upon compressions due to electrostatic attractions.

Prof. C. H. **LEES** remarked that if the action of the coherer depended upon thermal effects, the sensitiveness would be affected by using materials of different thermal conductivities, whereas if the effects were to be attributed to compressions, change of material would have little effect upon the sensitiveness.

Mr. **RAYNER** pointed out that if the effect was due to the temperature coefficient of the resistance of the film, the sensitiveness should depend upon the temperature at which the coherer was worked. He asked if Dr. Eccles had conducted experiments at different temperatures and if he could give any particulars as to how the sensitiveness of the coherer he had used compared with other types.

Dr. **RUSSELL** suggested that as the thickness of the film of oxide on the iron plate was only about a micron, the electrostatic attraction between the needle and the iron plate would be appreciable even at very low voltages. It was conceivable that, owing to the compression, heat was generated in the film at this point.

Mr. **CAMPBELL** remarked that in accurate work there would be no trouble in measuring the mutual inductances, but the determination of the damping factors would be difficult.

Dr. **ECCLES**, in reply to Mr. Duddell, said that the hypothesis that electrostatic attractions between the two conductors separated by the film of oxide caused the change of conductivity produced by electrical oscillations, was negatived by the fact that in such materials as the Author had tried the magnitude of the coefficient of decrease of resistance with temperature and the thermoelectric properties of the two conductors had a great deal to do with the sensitiveness of the detector. In reply to Prof. Lees, no doubt the sensitiveness of a detector would, if the thermal theory were correct, depend upon the slowness with which the heat was conducted away by the metal near the contact. This consideration suggested one reason why some of the badly conducting oxides formed such good detectors. With regard to Mr. Rayner's observations, he had not attempted experiments at other than normal temperatures. The Author, replying to Dr. Russell, said the answer to Mr. Duddell was circumvented by his suggestion that the electrostatic attractions caused compressions which on the whole produced local heating. But it might be pointed out that if in the secondary circuit the capacity was changed to another value and the inductance altered to keep the period of the circuit the same, the oscillating voltage at the coherer was different, yet no difference worth mentioning was heard in the detector. In answer to Mr. Campbell, no doubt the mutual inductance could be measured accurately given proper experimental facilities. Since, however, another course of experiments was adopted, one that did not need an accurate value of the mutual inductance between primary and secondary, this measurement was not attempted accurately.

XXI. *Earth-Air Electric Currents.* *By* GEORGE C. SIMPSON, *D.Sc., Government of India Meteorological Department* [*].

CONSIDERING the present state of our knowledge of atmospheric electricity, perhaps the most important factor requiring study is the current of electricity which is constantly passing between the earth and the air. This current has already been investigated to some extent, and two methods have been used for the purpose. Gerdien [†] determined the current by measuring the potential gradient and the conductivity of the air; and Wilson [‡] has investigated it by observing the actual loss of electricity from a small plate exposed to the atmospheric field and kept at zero potential. The measurements made by these two methods have only been few; and in view of the large variations which the current undergoes in the course of a few minutes, the need has been strongly felt for some method by means of which the current could be automatically recorded.

Neither Wilson's nor Gerdien's methods lend themselves for automatic registration, but an instrument has been developed in the Meteorological Office, Simla, which has proved entirely satisfactory in use. If we confine ourselves to the consideration of normal conditions during fine weather, the problem may be stated thus:—

Consider a small area of the earth's surface. On account of the electrical field in the atmosphere there is an induced surface charge of negative electricity on the area. This charge suffers change on account of two causes: (1) Every change in the potential gradient is accompanied by a change in the induced charge—the greater the potential gradient the greater the charge, and *vice versa*. (2) The electrical field in the atmosphere is constantly driving free positive ions from the air on to the surface; or, as it is generally expressed, the conductivity of the air results in a current of negative electricity from the earth into the air.

As the area under consideration forms part of the surface of the earth, its potential remains zero; hence each change of charge from either of these causes gives rise to a flow of electricity between the area and the earth. If, on the other hand, the area were insulated its potential would be constantly undergoing change unless steps were taken to keep it zero, by either adding or withdrawing a charge similar in amount to that which would have flowed to or from it if it had been earth connected. The problem therefore consists in finding a method for keeping an insulated plate placed on the ground at zero potential, and for measuring the change of charge which this entails.

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[*] Read March 11, 1910.

[†] Gerdien, *Götting. Nachr.* 1907, p. 77.

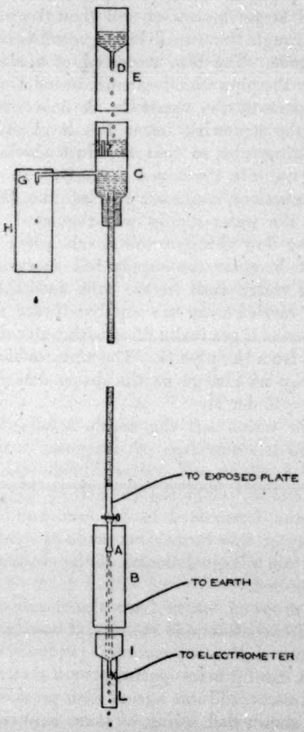
[‡] C. T. R. Wilson, Proc. Roy. Soc. A. lxxx. p. 537 (1908).

The solution obtained was the following:—Water was made to flow from an insulated vessel in fine jets, and the points at which the jets broke into drops were surrounded by an earth-connected cylinder. The vessel was connected by a wire to a large plate placed in the open as near to the ground as was consistent with efficient insulation. By the well-known "collector" action the jets prevented the potential of the insulated system from varying appreciably from zero, and the drops as they formed carried with them the charge necessary to effect this. In other words, the drops carried away all the charge set free. on the insulated plate which remained at zero potential. The drops as they fell were caught in an insulated vessel connected to an electrometer, and the change in the deflexion of the electrometer recorded the charge carried away by the drops.

The details of the apparatus as used in Simla are shown diagrammatically in fig. 1. The nozzle from which the water issued is shown at A, surrounded by the earth-connected cylinder B. In order to get a large discharging capacity with a small expense of water, it was necessary to have the water issuing from the nozzle in fine jets which broke up into very small drops. For this reason, the nozzle was attached to the lower end of a glass tube 150 cms. long which was usually kept full of water, so that the jets issued under a considerable pressure. There were three jets each having a diameter of about ·2 mm. To the top of the glass tube a vessel C was attached into which a regulated supply of water dropped from the funnel D. The point at which the water detached itself from the funnel D was well within the surrounding cylinder E, so that the drops were entirely free from charge when they fell into the vessel C.

At first considerable trouble was experienced on account of the fine holes in the nozzle becoming choked up. To prevent this filter-papers were introduced into the vessel C and also just above the nozzle itself; but it was found that

Fig. 1.



owing to some form of impurity in the water, the papers became impervious to water after being in use for a day or so. After trying several methods of filtering the water without success, the following satisfactory method was hit upon. A piece of sponge was placed in the funnel D to remove all large impurities, and then the water was caused to pass through the trap F in the vessel C before it reached the glass tube. The trap consisted of a closed brass tube fitting over the pipe which communicated from the upper to the lower parts of the vessel C. A hole was made in the covering tube somewhat below the level of the top of the communicating tube, so that the dust which floated on the top of the water in the upper part of C, and the grit which sank to the bottom, could not pass into the glass tube.

Usually the water-supply was regulated so that it just provided the flow through the nozzle when the glass tube was full; if, however, the supply fell short of this amount, the head of water sank in the tube until the flow from the nozzle just carried away the supply; if the supply was too great, the vessel C gradually filled with water until the surplus overflowed from the pipe G. The water which escaped here carried away no charge as the drops detached themselves within the cylinder H.

The water which left the nozzle A fell into the insulated funnel I and left this free of charge by dropping from the end of the small pipe K well within the cylinder L. Thus all the electricity which the drops from the nozzle carried into the funnel I remained in the funnel and its connexions. The capacity of this funnel was made as small as possible in order to obtain a large deflexion of the electrometer to which it was connected.

As the drops of water from the nozzle entered the receiver I with considerable velocity, it was deemed necessary to investigate whether any error was produced by the "Lenard effect." A careful investigation proved that no error of this kind was present, and this agreed with previous experiments, which had shown that, owing to some impurity in the Simla tap-water, it is incapable of developing the Lenard effect. If this work were repeated elsewhere it would be necessary to test for this source of error, and if found it would no doubt be possible to remove it by adding a little salt to the water.

The electrometer used was one of the pattern designed by Benndorf for automatically recording the potential gradient. In this instrument an electrical circuit containing a magnet is closed every two minutes, and a light boom attached to the needle of the electrometer is thereby pressed into contact with the paper taking the record. Thus the deflexion of the freely swinging electrometer-needle is automatically recorded at the end of each two minutes. A slight change in the construction of the electrometer made it possible to automatically earth the receiver as soon as its potential had been recorded. Each deflexion, therefore, was proportional to the charge obtained by the receiver in the previous two minutes.

The potential gradient was recorded by a second Benndorf electrometer, and the recording mechanisms of both electrometers were actuated by the same electrical current. Thus simultaneous records were made on the two instruments.

It was desirable that the exposed plate should be placed where the earth's normal electrical field was undisturbed. This, however, was quite impossible in Simla, for the town is situated on a ridge of the Himalayas, and there is no single plot of ground within many miles of the town on which the field is not disturbed by the configuration of the surrounding hills and valleys. The Simla Meteorological Office is at present located in a house originally built for a dwelling on a spur branching off the main ridge on which the town is built. The house is situated on the extremity of the spur, and the ground slopes down for many hundreds of feet at a very steep incline on three sides of it. There is, however, between the house and the main ridge a small piece of level ground (about 100 ft. × 50 ft.) on which a tennis court has been made. The tennis court is surrounded by stop-nets about ten feet high and several trees 30 or 40 feet high are growing against them. The branches of the trees which were growing over the court have been cut away so that the court itself is fully exposed to the sky. The middle of this court was the best site which could be obtained, and a small tea-house built close to the stop-nets on one side afforded a suitable shelter for the instruments. Thus the plate was exposed in the middle of a small open space, the electrical field of which was considerably affected by the near trees and stop-nets as well as by the irregularities of the surrounding hills.

The exposed plate itself consisted of a wooden frame (285×570 cms. $=17$ metre^2) on which canvas was tightly stretched. The canvas was covered by a sheet of brown paper made conducting by means of a thick coating of blacklead. In this way a large conducting surface was obtained which was at the same time light and strong. The sheet was insulated on six sulphur insulators which raised the upper surface of the sheet about 15 cms. above the general level of the ground. It would have been much better if the sheet could have been actually at the ground level, but as this would have necessitated cutting up the surface of the tennis court it was not possible to do it. It will be seen from the above that the exposure of the plate was far from that of a piece of the earth's surface exposed to the normal potential gradient; but it was the best which could be obtained under the circumstances.

Results.

Let each cm.^{2} of the exposed plate be receiving a charge from the air at the mean rate of x els. units per second. Then in a given time t the whole plate will receive from the air Axt els. units. If the potential gradient immediately over the plate changes from P_1 to P_2 volts/metre during the interval, there will be also a quantity of electricity $-\frac{A(P_1-P_2)}{4\pi 30000}$ els. units set free on the plate. Both these charges are transferred to the receiver; hence, if the mean charge per second recorded by the electrometer is Q els. units, we have

$$Qt = Axt - \frac{A(P_1-P_2)}{4\pi 30000},$$

or

$$x = \frac{Q}{A} + \frac{P_1-P_2}{t\,.\,3{\cdot}77\times 10^5}\text{ els. units.}$$

If it is assumed that the whole charge x reaches the plate by means of the ordinary conductivity of the air (*i. e.*, neglecting the charge carried by dust, &c. which settled on the plate against the electrostatic field), the conductivity can be calculated from the relationship

$$\lambda = x \Big/ \frac{P}{30000} \text{ els. units,}$$

in which P is the mean potential gradient expressed in volts/metres during the interval t. This mean potential gradient will in general be different from $\frac{1}{2}(P_1 + P_2)$.

This treatment applies to a plate in a uniform electrical field, and therefore will not hold strictly for the plate exposed in the disturbed field described above. It has, however, been applied, and in discussing the results it must be remembered that while the absolute values obtained are likely to be affected, the daily range of the different factors are no doubt near approximations to the truth.

The data obtained were treated thus :—The trace printed by the electrometer consisted of a series of dots, each representing the deflexion at the end of a two minute interval. These dots were first joined by a line, and then a freehand curve was drawn to take out the smaller irregularities.

The approximately smooth curve thus obtained was measured at five equidistant points in each hour, and from these five measurements the mean deflexion of the electrometer during the hour determined. The mean deflexion was then converted into volts and the quantity Q/A determined. The part of this value due to the change in the potential gradient was then obtained from the actual value of the potential gradient at the beginning and the end of the hour. The latter charge was subtracted from Q/A, and the remainder gave the mean value of x during the hour.

The following gives a typical example :—The smoothed curve for the hour 20 to 21 Indian standard time on the 19th November, 1909, was measured at five equidistant points, giving a mean value of 30·8 mm. As the sensitiveness of the electrometer was 5·9 volts per mm. deflexion, the mean voltage recorded at the end of each two minutes during the hour was $30{\cdot}8 \times 5{\cdot}9 = 181{\cdot}7$ volts, *i. e.* ·606 els. unit.

TABLE I.

Indian Standard Time.		0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10	10 to 11	11 to 12	12 to 13	13 to 14	14 to 15	15 to 16	16 to 17	17 to 18	18 to 19	19 to 20	20 to 21	21 to 22	22 to 23	23 to 24	Mean of day.	
17th Nov., 1909.	x …	75	63	62	48	43	42	38	37	45	50	49	48	59	58	49	46	50	48	54	64	71	71	74	64	54	×10^{-8} els.unit.
	P.G. …	26	26	26	25	17	16	11	14	14	32	37	34	40	39	32	32	32	37	40	42	40	36	32	33	30	Volt/metre.
	λ …	8·4	7·2	6·9	6·0	7·5	8·1	10·8	7·8	9·9	4·8	3·9	4·2	4·5	4·5	4·5	4·2	4·8	3·9	4·2	4·5	5·4	5·9	7·2	5·7	6·0	×10^{-4} els.unit.
19th Nov.	x …	62	75	62	49	43	42	40	47	52	54	52	45	52	61	63	49	49	52	48	74	102	80	83	84	58	×10^{-8} els.unit.
	P.G. …	33	28	22	14	18	22	18	14	27	45	36	30	34	34	36	32	41	37	37	31	35	33	24	30	30	Volt/metre.
	λ …	5·7	8·1	8·4	10·5	7·2	5·7	6·6	10·3	5·7	3·0	3·6	4·5	4·5	5·4	5·1	4·5	3·6	4·2	3·9	7·2	8·7	7·2	10·4	8·4	6·4	×10_{-4} els.unit.
20th Nov.	x …	75	75	58	42	42	39	33	47	59	61	64	71	64	55	52	50	46	42	49	67	68	72	70	65	57	×10^{-8} els.unit.
	P.G. …	36	27	32	32	27	23	26	40	35	32	36	37	35	33	31	34	39	40	46	39	30	30	33	35	34	Volt/metre.
	λ …	6·3	8·1	5·4	3·9	4·5	5·1	3·9	3·6	5·1	5·7	5·4	5·7	5·4	5·1	5·1	4·5	3·6	3·0	3·2	5·1	6·9	7·2	6·3	5·4	5·1	×10^{-4} els.unit.
21st Nov.	x …	73	64	61	48	42	38	41	33	44	52	50	59	64	60	49	40	41	51	60	52	56	64	55	53	52	×10^{-8} els.unit.
	P.G. …	37	30	22	22	17	17	21	20	32	33	32	34	34	32	28	28	26	31	28	32	26	27	29	23	28	Volt/metre.
	λ …	6·0	6·3	8·3	6·6	7·5	6·6	5·8	4·8	4·2	4·8	4·8	5·1	5·7	5·7	5·2	4·2	4·8	4·8	6·3	4·8	6·3	7·2	5·7	6·9	5·8	×10^{-4} els.unit.
22nd Nov.	x …	38	45	48	45	46	41	40	37	50	56	51	54	63	48	45	47	49	59	50	59	65	62	60	60	51	×10^{-8} els.unit.
	P.G. …	31	31	26	24	23	22	21	20	20	21	24	29	31	28	27	27	28	31	35	34	35	31	33	37	28	Volt/metre.
	λ …	3·6	4·2	5·7	5·7	5·9	5·4	5·8	5·7	7·2	7·8	6·3	5·7	4·2	5·1	5·0	5·2	5·3	5·8	4·2	5·1	5·7	6·0	5·4	4·8	5·5	×10^{-4} els.unit.
24th Nov.	x …	54	67	59	51	57	54	43	40	37	39	54	63	61	48	47	43	44	44	50	53	48	57	44	54	50	×10^{-8} els.unit.
	P.G. …	40	35	25	30	24	24	28	33	33	35	36	41	36	33	34	32	31	37	36	28	29	32	31	30	32	Volt/metre.
	λ …	4·2	5·7	7·2	5·1	7·2	6·6	4·5	3·6	3·3	3·3	4·5	4·5	5·2	4·4	4·2	3·9	6·0	3·6	4·2	5·7	4·8	5·1	4·3	5·4	4·9	×10^{-4} els.unit.

25th Nov.	x	…	54	52	45	44	51	46	46	37	34	43	50	48	46	46	39	38	48	53	48	52	61	60	64	60	49	$\times 10^{-8}$ els.unit.
	P.G.	…	28	20	21	16	17	17	16	11	17	24	26	24	24	25	21	22	23	24	25	26	24	23	22	22	22	Volt/metre.
	λ	…	5·7	7·8	6·3	8·4	9·0	7·8	8·7	9·9	5·7	5·4	5·7	6·0	5·7	5·7	5·4	5·4	6·3	6·7	5·9	6·0	7·6	7·8	8·7	8·1	6·9	$\times 10^{-4}$ els.unit.
27th Nov.	x	…	50	48	44	42	51	37	57	56	65	62	54	51	50	40	45	40	38	56	47	48	46	52	67	71	51	$\times 10^{-8}$ els.unit.
	P.G.	…	32	32	34	26	26	22	19	21	20	24	22	29	27	26	26	26	28	28	27	25	27	20	20	21	25	Volt/metre.
	λ	…	4·8	4·5	3·9	4·8	6·0	5·1	9·3	8·1	9·9	7·8	7·2	5·4	5·4	4·8	5·4	4·8	4·2	6·0	5·1	5·7	5·1	8·0	10·2	10·2	6·3	$\times 10^{-4}$ els.unit.
28th Nov.	x	…	68	74	65	65	54	52	53	50	64	74	57	61	56	37	47	40	42	54	55	56	52	36	54	45	55	$\times 10^{-8}$ els.unit.
	P.G.	…	21	21	21	22	23	22	22	24	26	23	33	31	28	27	25	27	27	28	21	38	39	33	34	34	27	Volt/metre.
	λ	…	9·6	10·5	9·3	9·1	7·1	6·9	7·4	6·3	7·2	6·6	5·1	6·0	6·0	4·2	5·7	4·4	4·5	5·7	7·8	4·5	3·9	3·3	4·8	3·9	6·4	$\times 10^{-4}$ els.unit.
30th Nov.	x	…	69	61	58	51	44	55	43	44	55	48	48	52	62	54	58	53	45	62	65	84	94	56	74	69	58	$\times 10^{-8}$ els.unit.
	P.G.	…	40	35	25	27	29	22	17	20	23	33	38	37	43	37	34	38	40	34	43	38	49	48	37	38	34	Volt/metre.
	λ	…	5·1	5·1	7·2	5·7	4·8	7·6	7·4	6·7	7·0	4·5	3·9	4·2	4·5	4·5	5·1	4·1	3·4	5·4	4·5	6·6	5·7	3·6	6·0	5·4	5·3	$\times 10^{-4}$ els.unit.
Actual mean values for ten days.	x	(67)	62	62	56	49	47	44	44	43	51	53	52	55	58	51	49	45	45	52	53	61	66	61	65	62	54	$\times 10^{-8}$ els.unit.
	P.G.	(31)	32	29	25	24	22	21	20	22	25	30	32	33	33	31	29	30	32	33	34	33	33	31	29	30	29	Volt/metre.
	λ	64	5·9	6·8	6·9	6·6	6·7	6·5	7·0	6·7	6·5	5·7	5·0	5·1	5·1	4·9	5·1	4·5	4·6	4·9	4·9	5·5	6·0	6·1	6·9	6·4	5·8	$\times 10^{-4}$ els.unit.
Mean values corrected for non-periodic change.	x	65	60	61	55	47	46	44	43	42	50	52	52	55	60	51	50	45	46	53	54	62	68	63	67	65	54	$\times 10^{-8}$ els.unit.
	P.G.	31	32	28	25	23	22	21	20	21	25	30	32	33	33	32	30	30	32	33	34	34	34	32	30	31	29	Volt/metre.
	λ	6·4	5·9	6·8	6·9	6·6	6·7	6·5	7·0	6·7	6·5	5·7	5·0	5·1	5·1	4·9	5·1	4·5	4·6	4·9	4·9	5·5	6·0	6·1	6·9	6·4	5·8	$\times 10^{-4}$ els.unit

The combined capacity of the receiver and electrometer was 35 cm.; hence, during this hour the average quantity of electricity removed from the plate during each two minute interval was $\cdot606 \times 35 = 21\cdot21$ els. unit, or during each second $21\cdot21/120 = \cdot177$ els. unit. The area of the plate was 17×10^4 cm.^{2}, hence $Q/A = \cdot177/17 \times 10^4 = 104 \times 10^{-8}$ els. unit.

At the commencement of the interval the potential gradient over the plate was 20·8 volt/metre and at the end 49·3 volt/metre, hence

$$\frac{P_1 - P_2}{t \, . \, 3\cdot77 \times 10^5} = -\frac{28\cdot5}{3600 \times 3\cdot77 \times 10^5} = -2 \times 10^{-8} \text{ els. unit.}$$

Hence x, the mean charge received from the air each second by each cm.^{2} of the exposed plate, was

$$(104 - 2)10^{-8} = 102 \times 10^{-8} \text{ els. unit.}$$

The mean potential gradient over the plate during the hour under consideration was 34·8 volt/metre, hence

$$\lambda = 102 \times 10^{-8} \Big/ \frac{34\cdot8}{30000} = 8\cdot7 \times 10^{-4} \text{ els. unit.}$$

The values obtained during ten days in November are shown in Table I. Only those days on which the weather was fine, without a cloud in the sky, and on which an uninterrupted record was obtained from midnight to midnight, have been utilized.

The lines marked x give the mean charge received from the air by each cm.^{2} of the exposed plate in a second during the hour stated at the head of the column.

The lines marked P.G. give the mean potential gradient over the exposed plate during the hour.

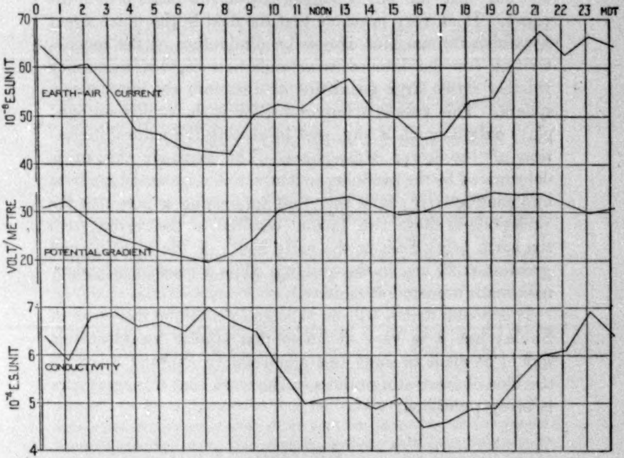
The lines marked λ give the mean conductivity of the air during the hour.

The time entered at the head of the table is Indian Standard Time, which is 21 minutes ahead of mean local time.

In fig. 2 the mean values for the ten days are plotted. In the figure the mean value during the hour is referred to the middle instant of the hour expressed in local time. Thus the value for the hour 10 to 11 I.S.T. is plotted against 10 hrs. 9 min. local time.

It has already been pointed out that the exposure of the plate was not satisfactory, so that the absolute values are uncertain; but it is interesting to note that the mean value found for the conductivity of the air, viz. $5{\cdot}8 \times 10^{-4}$ els. unit, agrees very well with that found by other workers, all of whom have found values of the order of magnitude of 10^{-4} els. unit: see the list given on page 83 of H. Mache and von Schweidler's book *Atmospherische Elektrizität.*

Fig 2.



Turning now to the daily variations which are much less affected by the unsatisfactory exposure, the curves show that the potential gradient had its chief minimum and maximum in the early hours of the morning and evening respectively. The conductivity was above the average during the night and early morning, and below during the day and evening. During the period over which the observations extended the sun rose above the surrounding hills soon after 7 hrs. local time and set at about 17 hrs. Thus it would appear as if the sunshine caused a diminution of the conductivity of the air, but that the effect lagged about two hours behind the cause. It is very probable that the dust is the chief agent by which the sunshine causes the diminution of the conductivity; for the intense solar radiation of the Indian dry weather raises large quantities of dust into the lower atmosphere. This result is important, for it shows that sunlight plays only a small, if any, part in causing the natural ionization of the lower atmosphere. The earth-air current is determined by the two independent variables potential gradient and conductivity; it is therefore interesting to note that the variations in these two factors resulted in the current from the earth being least in the early hours of the morning and greatest at 20 hrs. in the evening, while a marked secondary maximum occurred at midday.

It is hoped that it will be possible to continue this work in Simla; but it is very desirable that similar measurements should be made in some less abnormal region, and it is with the idea of interesting others in the work that this short note is being published.

ABSTRACT.

The paper describes a method for automatically recording the electrical current which passes from the earth into the air during periods of fine weather.

A large plate (17 metres^2) was placed in the open as near to the ground as was consistent with efficient insulation, this was then connected to an insulated vessel from which water issued through an orifice surrounded by an earth-connected cylinder. The water as it dropped from the insulated vessel carried away, by the well-known "collector" action, all the charge which the exposed plate received and the latter remained at zero potential. The charged water drops were collected in a vessel connected to a self-registering electrometer which was earth-connected for an instant at the end of every two minutes. The paper describes the sources of error and the method of determining the value of the earth-air current and of the conductivity of the air from the records of the electrometer. The method was used in Simla (India), but

owing to the impossibility of obtaining a site on which the normal electrical field of the atmosphere was undisturbed by the surrounding hills and trees, the absolute values obtained were uncertain, but it is very probable that the daily range of the different factors were near approximations to the truth.

DISCUSSION.

Dr. C. **CHREE** expressed his interest in the paper and remarked that they had worked at Kew with C. T. R. Wilson's apparatus. It would be interesting to know whether the same results could be obtained from the Author's and from Wilson's apparatus. He drew attention to the difficulty of insulating a piece of the earth in England sufficiently well to obtain reliable results.

Dr. **RUSSELL** said that C. T. R. Wilson had found that the mean earth leakage current in his experiments was about $2{\cdot}2\times10^{-16}$ amperes per square centimetre. It was therefore exceedingly small.

Linke had shown from the results obtained in balloon ascents that the atmospheric potential gradient diminished regularly up to a height of 6000 metres and probably much higher. Assuming that air has "conductivity" this shows that the conductivity increases as we ascend. It seems reasonable to suppose that at great altitudes, notwithstanding the intense cold, the highly rarefied air is practically a conductor. The earth leakage currents carry charges to these conducting layers, and these charges are continually being returned to the earth by the storms always taking place somewhere or other in the world. The energy expended by the currents was probably due originally to the sun's heat vaporizing and raising water to heights in the air. A surprising result obtained by Dr. Simpson was that the sunshine apparently caused a diminution in the conductivity of the air over the surface of the ground. It was customary to suppose that ultra-violet rays improve the conductivity of the atmosphere, and this supposition was certainly a help in explaining how radio-telegraphic waves went further over the ocean by night than by day.

XXII. *An Automatic Toepler Pump, designed to collect the gas from the apparatus being exhausted.* By BERTRAM D. STEELE, *D.Sc.*[*]

DURING the investigation of certain dissociating compounds containing sulphur dioxide, it became necessary to exhaust constantly certain apparatus for a prolonged period. After several days had been spent in manipulating a Toepler pump of the usual pattern, an automatic Toepler pump was designed and constructed.

The pump that was required was one which not only should be capable of being worked automatically for long periods, but also one by means of which it should be possible to deliver samples of the extracted gas when desired, and if necessary to collect the whole of such gas for examination.

These requirements are satisfactorily fulfilled by the pump which is described in the following paper. It will be best described in three sections, all of which are shown diagrammatically in fig. 1 (p. 327).

The pump proper is an ordinary Toepler pump, and is represented in the figure by the parts A and B and various adjuncts.

The collecting apparatus is represented by the parts C and D and connecting tubes.

The automatic controlling apparatus is represented by *s*, F, E, and *q*.

The pump proper consists of the stroke-cylinder A and the reservoir B, these being connected by the U-tube *d*. This U-tube, which is about 8 mm. in internal diameter, is provided at its lowest point with a short side-tube, by means of which the mercury can be removed from the pump if necessary, and it is constricted at one point to a diameter of about 2 mm. This constriction regulates the flow of mercury and renders it impossible to break the apparatus by a sudden blow of the mercury, even at the highest exhaustion. The U-tube in its shorter arm is 800 mm. long, in order that either A or B may be put into communication with the atmosphere whilst the other vessel is exhausted.

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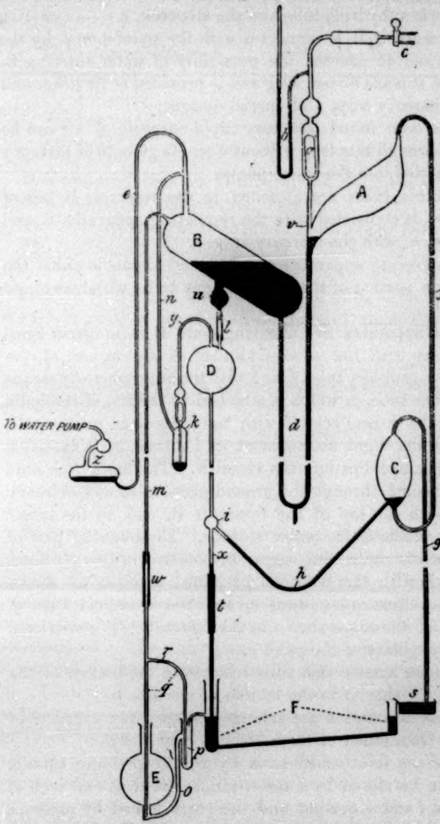
[*] Read March 11, 1910.

Fig. 1.



The stroke-cylinder A is provided, in the usual manner, with a ground-glass valve, *a,* and a manometer, *b,* and the apparatus to be exhausted is attached on the other side of the phosphoric-anhydride tube and the stopcock, *c*.

The reservoir B is connected with the water-pump by the tube *e*, and to prevent the possibility of water entering B, this tube is made 85 cm. long and is provided at its lower end with a mercury trap, *z*, of special design.

It has been found that very rapid currents of air can be sucked through this trap without a single globule of mercury being carried into the water-pump.

Two other tubes are attached to the reservoir B, one of which, *m*, is connected with the regulating apparatus E, and the other, *n*, with the mercury trap, *k*.

The collecting apparatus was designed to enable either the whole or a portion of the extracted gas to be withdrawn and collected.

In this apparatus the receiving bulb C is in direct communication with the stroke-cylinder A by means of the millimetre capillary tube, *f*, and with the atmosphere by means of a similar tube, *g*, which is attached to the top of the bulb. The U-tube *h* connects C with the subsidiary reservoir D, and from the slight enlargement at *i* a tube, *x*, is extended downwards and dips into the vessel F. The top of the bulb D is connected through the ground-glass valve and mercury trap *k* with the top of the reservoir B, and by the tube *l* with the middle of its under surface. The essential part of the automatic controlling apparatus consists of two portions, the bulb E with the test-tube *p*, joined together as shown by the 8-millimetre bore-tube *o*, and the branched tube *q*, which is of the shape shown in the figure, and is constricted to a fine capillary at the point *r*.

The longer arm of this tube reaches to the bottom of the vessel E, the shorter to the bottom of the tube *p*.

The vessels *s* and F are arranged so that any overflow of mercury from either of them finds its way into *p*.

The tube *q* is attached to *m* by rubber pressure tubing, which can be closed by a screw pinchcock at *w*, and each of the tubes, *l* and *x*, are cut and the parts joined by pressure tubing, which in the same manner can be closed by screw pinchcocks.

The volumes of the various vessels are approximately as follows :—

The stroke-cylinder	A	 300	cubic	centimetres.
The reservoir	B	 600	,,	,,
The collector	C	 100	,,	,,
The reservoir	D	 130	,,	,,
The bulb	E	 150	,,	,,

It is essential that the U-tube *d*, and the tubes *e*, *f*, and *g* should be over 80 centimetres in length.

The relative levels of the various bulbs are as follows :—

The bottom of the reservoir B is placed 30 centimetres below the top of the tube *f*, and 10 centimetres below the connexion between A and the valve *a*, at *v*. The bottom of the reservoir D is situated 16 centimetres above the highest point of the tube *g*.

The shorter limb of the U-tube *h* is 25 centimetres long, and the enlargement *i* is 75 cm. above the level of the side-delivery tube on F.

Twenty pounds of mercury are required to work a pump, of the foregoing dimensions.

Filling the Pump.

Before starting to fill the pump, the stopcock *c* and the pinchcock on *l* are closed, and mercury is poured into the vessels *s* and F, and into the traps *z* and *k*. The trap *k* is provided with a side-tube at the point *y*, to enable this to be done, and when sufficient mercury has been inserted this tube is sealed off. The water-pump is now connected at the trap *z*, and the mercury required to fill the pump is poured into E. Under the action of the water-pump, which must be sufficiently powerful to reduce the pressure in the apparatus to 10 or 15 cm. (whilst the side-tube is open to the atmosphere), the mercury rises in the tube *m*. Air is, however, slowly leaking in through the constricted branch of *q*, and this air breaks the mercury in *m* into columns which are carried over into B. In this way the whole of the mercury required is pumped rapidly from E into B.

Working the Pump.

It has been already stated that the pump was designed (1) to discharge the extracted gas into the water-pump, (2) to collect samples of the gas for examination, and (3) to collect the whole of the gas from any piece of apparatus. The manner of working of the pump will be best shown by describing in detail the course of one complete stroke when it is working as required for case (1).

Starting from the time when a small quantity of the mercury required for filling the pump remains in E, and when the pressure in A, B, C, and D has been reduced to 10 or 15 centimetres, it will be seen that with the removal of the last globule of mercury from E into B, air can freely enter the latter vessel through the tube *m*. The water-pump is not capable of removing the air as quickly as it enters, and the consequent rise in pressure causes the mercury to rise in A, and, driving the gas before it into the receiver *c*, it overflows through the tubes *h* and *t* into the vessel F, from this into *p*, and finally through *o* into the large bulb E. This overflowing mercury closes first the constricted and then the open end of the branched tube *q*, and thus cuts off the supply of air to the reservoir B.

The water-pump now reduces the pressure in the reservoirs B and D, and as a consequence the mercury immediately begins to flow back through the U-tube *d*, from A into B, and the overflow into F and *p* ceases.

About forty seconds after the closing of the tube *q* the mercury will have fallen to the point *v*, and in the meantime the mercury in *p* has been slowly siphoning through the branched tube *q* into the bulb E.

With the constriction used, the time required to empty *p* is eighty seconds, and there is therefore an interval of forty seconds between the opening of the communication at *v* and the emptying of the tube *p*. During this time the apparatus to be exhausted remains in communication with the stroke-cylinder A. Following the last globule of mercury which siphons from *p* into E, air enters through the shorter branch of *q*, and the mercury in E is pumped back into the reservoir

B, and finally, when all the mercury has been removed from E, air enters again freely through the tube *m*, and another stroke begins. The gas that accumulates in C finds its way through the tube *h*, the bulb D, and the trap *k*, to the water-pump.

Case (2). If it is desired to collect samples of the gas, this can be done as follows :—

A test-tube of the size required is filled with mercury and inverted over the turned-up end of the tube *g*, the pinchcock *t* is closed, and at the following stroke of the pump mercury flows, by way of *f*, C, and *h*, into the bulb D. When a quantity more than sufficient to fill the bulb C has accumulated here, the pressure in D is caused to rise by opening the pinchcock on the tube *l*, when the gas in C is forced into the tube placed to receive it. The excess of mercury flowing into *s* finds its way as before into *p* and E.

Finally the pinchcock *l* is closed, that at *t* opened, and the cycle proceeds as previously described.

Case (3). Should it be desired to collect the whole of the gas from the apparatus being exhausted, the pinchcock *t* must be kept closed, and a vessel to receive the gas must be placed as before over the end of *g*. If the pressure in the apparatus is low, that is, if the quantity of gas to be collected is very small, the gas is displaced from C at every stroke of the pump, and the overflowing mercury finds its way from *s*, through F, into *p* and E.

If the pressure is more than 1 or 2 centimetres, it is necessary to manipulate the pinchcock on *l*.

In the great majority of instances it is only during the first five or six strokes of the pump that this will be necessary, and in this case it is most convenient to open *l* at the end of each stroke, exactly as described in case (2.)

Should the vessel to be exhausted be very large, the pinchcock *l* must be regulated so that the mercury contained in the bulb *u* flows into D at such a rate that the last drop, followed by air, enters D at the moment when the latter has become about half full of mercury. If *l* is properly regulated, it is found that mercury flows through it only when the tube *m* is open to the atmosphere and not when the reservoir B is exhausted by the water-pump.

In obtaining high exhaustion it is advisable to leave the stroke-cylinder A in open communication with the apparatus to be exhausted for a longer time than 40 seconds. This object is obtained by attaching a side-tube to *p*, the capacity of which increases the volume of mercury which must siphon through *q* at every stroke.

The quantity of mercury, and hence the time of flow, can be regulated by sliding a well-fitting glass piston in this subsidiary tube.

The time occupied by a complete stroke of the pump that is described here is 160 seconds, during 40 of which the communication at *v* is open. By the use of the subsidiary tube on *p* (not shown in the diagram) the total time is increased to four minutes, that during which *v* is open is increased to two minutes.

In order to stop the pump, all that is necessary is to close the pinchcock *w* when the tube *m* contains mercury, and then to disconnect the water-pump at *z* when the vessel A is empty of mercury. To start the pump afresh, the pump is attached to *z* and the pinchcock *w* is opened.

Chemical Laboratories,
The University of Melbourne.

XXIII. *Further Tests of Brittle Materials under Combined Stress.* *By* WALTER A. SCOBLE, *A.R.C.Sc.*, *B.Sc.*, *Whitworth Scholar* [*].

Introduction.

THERE are three theories of elastic strength which are recognized by engineers,—that a material fails under a definite (*a*) Maximum principal stress, (*b*) Maximum shear stress, (*c*) Maximum strain. A large number of tests on ductile metals have been made, and these have shown that the maximum shear stress approximately determines the failure of such materials, and consequently that theory is justified. Although tests in pure tension and torsion indicate that ductile materials fail by shearing, whereas brittle materials fracture across the plane of maximum principal stress, it has not been usual to make any distinction between the metals when the results of tests under combined stress have been recorded, or, at most, following the lead of Guest, the metals have been termed ductile. Brittle materials have been almost entirely neglected. Yet there are two important theories which have been disproved for ductile materials, which lead to very similar results, and which are supported by the forms of the fractures obtained with brittle materials. Even without any new experimental evidence, it appears to be very probable that brittle materials obey the maximum principal stress law.

Previous Tests.

The author used cast iron bars for his original tests of a brittle material under combined stress[†]. Fracture is the most satisfactory criterion of strengh for a brittle material. But it is doubtful whether cast iron follows Hooke's law with sufficient closeness to allow the ordinary elastic theories to be applied to it ; certainly it is not perfectly elastic to fracture. Consequently it is not a satisfactory material from the standpoint of an elastician, but it was adopted because it is the brittle material which is most commonly used in engineering practice.

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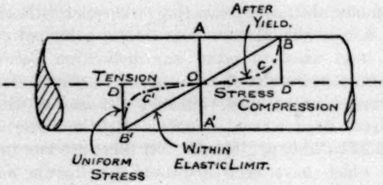
[*] Read April 22, 1910.

[†] Proc. Phys. Soc. London, vol. xx.

The results of the tests on cast iron showed that both the maximum principal and shear stresses at fracture varied considerably if they were calculated on the assumption that the metal was elastic to rupture. But the planes of failure approximated very closely to those of the maximum principal stress. The metal certainly yielded somewhat, and it was not possible to find the exact stress distribution, but it appeared to be reasonable to assume that the yield was sufficient to cause the stresses which were directly due to the bending and torsion to become uniform. For example, fig. 1 represents the centre of a bar under a central bending load. AOA′ is the section midway between the supports. The tensile or compressive stress (*at a point in the section*), due to the bending, is set out perpendicularly to AA′. The compressive stress at A is represented by AB. When the material has not been stressed at A or A′ beyond its elastic limit, BOB′ indicates the stress distribution over the section. If the material has yielded at the outside, the inner layers are more fully stressed, and a line like BCO replaces the straight line BO. A small yield causes BCO, which represents the true compressive stress at any point, to follow BDO more closely than OB. But BD represents a uniform stress over the section, and this condition was therefore assumed. The assumption that the shear stress due to torsion was uniform may be similarly justified.

Fig. 1.



The Specimens for the Present Tests.

The behaviour of cast iron, therefore, appeared to support the maximum stress theory, but it was necessary to assume that the known yield caused a redistribution of stress. This was not entirely satisfactory, and further to test this law it was decided to use a material which was elastic to fracture, or was more truly brittle, and which therefore obeyed the usual laws of stress distribution at rupture. Specimens of brittle alloys were obtained, but were condemned after a careful examination, without being tested under combined stress. The metal finally selected was Vickers, Maxim "Cast 7" steel, which contained 0·7 per cent. of carbon. The total length of a bar was 36 inches, and it was ¾ inch diameter, with ¾ inch squared ends. In order to obtain a material which was perfectly elastic to fracture it was necessary to harden the bars. Unfortunately a suitable furnace was not available for heating them. The assistance of an experienced toolsmith was obtained, and a spread fire was arranged on a smith's hearth. A long iron tube was heated as uniformly as possible in the fire over a greater length than that of a bar. A bar was placed inside the tube, and was frequently moved along it, and rotated, further to ensure uniform heating. When the bar had attained a suitable temperature, about 1400° Fahr., it was removed from the tube and lowered into dilute vitriol. The bar was kept vertical in order to keep it straight; it was washed later in strong soda water to neutralize the acid from the hardening bath. The temperature of a bar before it was quenched was not measured, it was only judged from its appearance. It was, therefore, unlikely that the bars were of equal strength after hardening, and it was decided to make more than one test on each bar, so that each specimen would afford independent evidence on the law of failure.

The Apparatus and Method of Testing.

The apparatus employed has been fully described elsewhere*. One end of a bar was supported so that it was free to take its natural slope under a bending load, but it was not allowed to twist. The bar rested on rollers at 30 inches from the other support, and therefore was free to twist at this end. A wooden pulley fitted on the squared end of the specimen and was twisted by means of ropes which carried dead weights. A bending load was applied at the centre of the bar. The maximum shear stress due to the torque was confined to the whole of the outer surface of a bar, and the bending load produced its maximum compressive stress only at the highest point, and its maximum tensile stress only at the lowest point, of the mid-section of the specimen. The test of a bar under combined loading was made first. For a further torsion test one part of the original bar was held in two hardened, serrated grips, or clamps, bolted together. The clamps were bolted to a lathe bed with the bar just resting on the roller bearing. The squared end of the half bar then carried the torsion pulley as before. When it was necessary to make a further bending test, a part of the original bar was supported on knife edges and loaded by weights applied at its centre.

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* Proc. Phys. Soc. London, vol. xx.; also Phil. Mag. Dec. 1906.

Calculation of the Stresses.

The maximum stress due to bending was calculated from the formula

$$\frac{p}{y}=\frac{\mathrm{M}}{\mathrm{I}},$$

in which

p is the maximum stress.

y is the greatest distance from the neutral line of the section $=\frac{d}{2}$.

M is the maximum bending moment acting on the beam.

I is the moment of the inertia of the section about its neutral line $=\frac{\pi d^4}{64}$.

d is the diameter of the bar.

The maximum shearing stress due to the torque was given by

$$\mathrm{S}=\frac{16\mathrm{T}}{\pi d^3}.$$

S is the maximum shearing stress.

T is the torque.

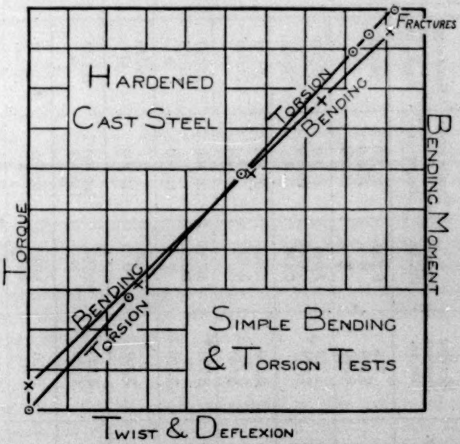
The maximum principal stress P_1 was $\frac{p}{2}+\sqrt{\frac{p^2}{4}+S^2}$, and the minimum principal stress P_3 was $\frac{p}{2}-\sqrt{\frac{p^2}{4}+S^2}$. The stress difference was $2\sqrt{\frac{p^2}{4}+S^2}$, and the maximum shear stress $\sqrt{\frac{p^2}{4}+S^2}$.

If the maximum principal stress be represented by P_1, and the least principal stress at the same point by P_3, then the maximum stress law states that $P_1=\text{constant}$; the maximum strain theory is represented by $P_1-\eta P_3=\text{constant}$, since in these tests the second principal stress, P_2, is zero; and the shear stress being constant corresponds to $P_1-P_3=\text{constant}$.

Data from the Tests.

Stress-strain diagrams for a pure torsion test and a simple bending test have been plotted in fig. 2. The diagram shows that under either kind of loading, stress and strain were

Fig. 2.



Hardened Cast Steel Bars. $\frac{3}{4}$ inch diameter. Effective length as a beam 30 inches.

Bar.	Bending Moment. lbs. in.	Torque. lbs. in.	Tensile stress due to Bending. lbs./sq. in.	Shear stress due to Torque. lbs./sq. in.	Maximum Principal Stress. lbs./sq. in.	Minimum Principal Stress. lbs./sq. in.	Stress difference = twice Max. Shear Stress.	Remarks.
1	935	6,200	12,720	75,400	82,060	−69,340	151,400	Low.
1	2,960	0	72,000	0	72,000	0	72,000	Correct.
1	3,570	0	86,850	0	86,850	0	86,850	High.
3	1,770	5,040	43,000	61,200	86,450	−43,450	129,900	Low.
3	2,880	0	70,000	0	70,000	0	70,000	Correct.
3	0	6,410	0	78,000	78,000	−78,000	156,000	Low.
4	2,350	3,940	57,150	47,900	84,350	−27,150	111,500	Correct.
4	0	5,860	0	71,250	71,250	−71,250	142,500	Low.
4	3,330	0	81,000	0	81,000	0	81,000	High.
9	2,360	4,750	57,400	57,700	93,200	−35,800	129,000	Correct.
9	0	6,280	0	76,400	76,400	−76,400	152,800	Low.
9	3,450	0	83,900	0	83,900	0	83,900	High.
12	0	6,020	0	73,200	73,200	−73,200	146,400	Correct.
12	3,090	0	75,100	0	75,100	0	75,100	High.
2	4,500	2,960	109,400	36,000	120,200	−10,800	131,000	Correct.
2	4,650	0	113,000	0	113,000	0	113,000	High.
6	2,170	1,660	52,750	20,200	59,600	−6,800	66,400	Correct.
6	0	4,190	0	50,900	50,900	−50,900	101,800	Correct.
11	2,215	0	53,850	0	53,850	0	53,850	Correct.
11	0	3,900	0	47,500	47,500	−47,500	95,000	Correct.

proportional to fracture, and therefore it was permissible to calculate the stresses from formulæ based on Hooke's law. It was not intended to compare the results from different bars, but the several tests on each particular specimen only. It was convenient, however, to divide the bars in three groups depending on their strengths.

The remarks were written against the data immediately after the tests were made, before the results were calculated and plotted. They are intended briefly to record observations made during the tests. For example, the bending tests were made in certain cases on twelve inch lengths, and therefore the bending moments recorded at fracture were probably high. In some of the pure torsion tests the bar fractured at two sections, and the test was marked correct. Twelve specimens were prepared, but four of them developed flaws in hardening.

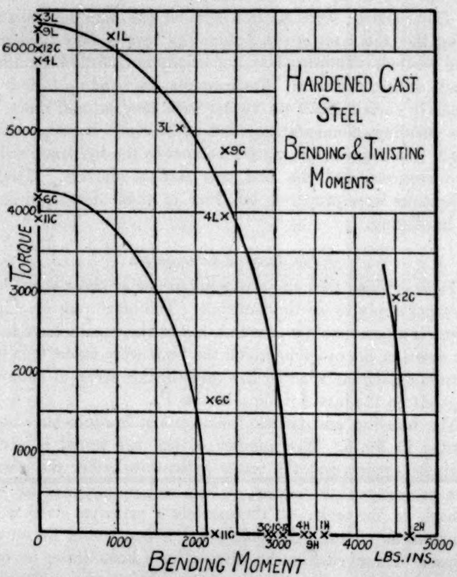
The Results Considered.

It is evident that the stress difference or shear stress law does not apply to brittle materials. The maximum principal stress is approximately constant, but as the results have been set down in the order in which the tests were made, it is not easy to determine whether the value of this stress at fracture depends on the least principal stress, P_3.

The bending and twisting moments at fracture have been plotted in fig. 3. The number of the bar tested is given against a point, and the letter indicates whether the result was probably high, correct, or low, and corresponds to the remark in the table. If the maximum principal stress were constant at rupture, the points for a bar would lie on an ellipse. Three such mean curves have been drawn in, one for each group of specimens. Considering the difficulties encountered in making the tests, the points lie fairly evenly about the curves. The deviation is most noticeable for the group which contains bars 6 and 11.

The author has expressed the opinion that the behaviour of all isotropic, homogeneous materials may be expressed in one form, that $P_1 - mP_3 = \text{constant}$, at the failure of the material, in which equation "m" is a constant whose value depends on the degree of ductility of the material*. P_2 does not appear in the equation because it is zero for the system of loading adopted, but even in the more general case, in which there are three principal stresses, the available

Fig. 3.



evidence indicates that the intermediate stress P_2 does not affect the values of P_1 and P_3 at failure, with the loadings which are met with in engineering practice. The advantage of the above equation is that when $m=0$, it represents the maximum principal stress hypothesis, when $m=\eta$, it indicates a constant maximum strain, and $m=1$ corresponds to the stress difference law. The maximum and the least principal stresses at rupture have been plotted in fig. 4. This least principal stress is the P_3 at the point at which the maximum principal stress operates, and although it has been called the minimum principal stress, it is not the algebraical minimum for the whole bar, but only at the point at which rupture commences.

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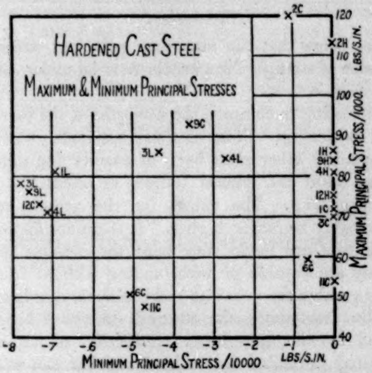
* Proc. Phys. Soc. London, vol. xxii.; also Phil. Mag. Jan. 1910.

Fig. 4.



The notation in fig. 4 is similar to that of fig. 3. It is easy to distinguish the three groups of tests. An increase of the minimum, P_3, does not regularly affect the maximum principal stress, P_1. The maximum strain is certainly not constant for a particular bar, and varies more than the maximum principal stress. Although the points are very irregular in this diagram, they are not more so than those usually obtained for a ductile material.

The angles between the surfaces of the fractures and the axes of the bars were measured, and agreed with those of the planes of maximum stress even better than in the case of the earlier tests on cast iron. For example, in a pure torsion test the angle which the plane of maximum stress makes with the axis of the bar is 45°. The corresponding angles measured from the fractured surfaces in five pure torsion tests were 43, 46, 44, 44, and 44 degrees. It was possible to measure the angles very accurately because the fractures were wonderfully clean, and as true as if the edges had been cut with a tool.

Conclusion.

The tests show that the maximum principal stress is the best criterion of strength for a brittle material under combined stress.

It is interesting to compare the strengths of the bars before and after hardening. None of these specimens were tested unhardened, but other steel bars, of exactly the same size, yielded at 2400 lbs. inches torque, or 2660 lbs. inches bending moment*. The values for the present material should probably be rather higher. It therefore appears that when the bars are made very hard the bending moment which they are capable of withstanding without fracturing is slightly greater than that at yield in the unhardened condition. But sometimes the strength to resist bending is diminished by the quenching. Similarly, as an average result, it may be stated that a hardened bar can withstand about double the torque which would cause it to yield when in its ductile state.

ABSTRACT.

A former paper described tests on cast iron, which is the brittle material which is most commonly employed in engineering practice. The cast iron yielded somewhat, and it was necessary to assume a redistribution of stress before fracture. Then the maximum principal stress appeared to be the best criterion of strength. The tests described in the present paper were made on hardened cast steel, which material was selected because it obeys Hooke's Law to fracture. The specimens were ¾ inch diameter and 30 inches effective length, and were tested under combined bending and torsion. As it was not possible to ensure that the hardened bars were of exactly equal strengths, a bar was first fractured under combined loading, then one portion was tested under bending alone, and the other part in simple torsion. Thus each bar afforded independent evidence on the law of failure.

Neither the maximum shear stress nor the maximum strain was constant at fracture, but the results indicated that the maximum principal stress is the best criterion of strength for a brittle material

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* Proc. Phys. Soc. London, vol. xx.; also Phil. Mag. Dec. 1906.

under combined stress. In general, the hardening did not affect the strength of a bar to resist bending, but it doubled the torque which was required to cause failure.

Discussion.

Dr. **Russell** asked the author for a definition of a *brittle* material.

Mr. R. S. **Whipple** asked what were the advantages of hardening in vitriol.

Mr. **Scoble**, in reply, said a brittle material was one which did not yield before fracture, although cast iron, which was usually classed as brittle, yielded to a slight extent. Tool hardeners found that hardening in vitriol was preferable to hardening in oil.

XXIV. *The Magnetic Balance of MM. P. Curie and C. Cheneveau. By* C. Cheneveau, *with an Appendix by* A. C. Jolley [*].

This apparatus is intended for the measurement of the coefficient of specific magnetization, or the susceptibility or permeability of feebly paramagnetic or diamagnetic bodies [†].

Principle and Theory of the Apparatus.

The body whose magnetic properties are to be determined is suspended from one end of the arm of a torsion balance. By means of this balance the force is measured, which is experienced by the body when placed in a non-uniform magnetic field, produced by a permanent magnet whose lines of force cross the space occupied by the body. The method of calculating this force will first be briefly indicated.

Suppose that the body is placed at a point O in a field of direction Oy and of intensity H_y. The force f which tends to move the body will be normal to the direction of the field,

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[*] Read April 22, 1910.

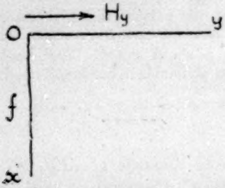
[†] The coefficient of specific magnetization K is the ratio of the intensity of specific magnetization $\mathcal{I}=\frac{M}{m}$ (where M is the magnetic moment and m the mass of the body) to the magnetizing field. The magnetic susceptibility $\kappa=K\Delta$, where Δ is the density of the body, and the permeability is obtained from the susceptibility by the relation $\mu=1+4\pi\kappa$.

i. e. in the direction Ox (fig. 1). If $\mathcal{I}$ is the intensity of specific magnetization, the value of the force is *

$$f=m\mathcal{I}\frac{\partial H_y}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\frac{\partial H_y}{\partial x}$ representing the space variation of the field.

Fig. 1.



As we are only concerned with feebly magnetic bodies, the demagnetizing force arising from the magnetization of the body is negligible, and we may assume that the intensity of magnetization is proportional to the field. If we denote the constant ratio between the intensity and field, or *coefficient of specific magnetization*, by K, we have

$$\mathcal{I}=KH_y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

consequently combining equations (1) and (2)

$$f=KmH_y\frac{\partial H_y}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let us first suppose that the magnet producing the magnetic field is at a considerable distance from the body. Then $H_y=0$, and by (3) the force is zero.

The body being always situated at O, let the magnet be brought up to the position I (fig. 2). If the force f is one of attraction the body is of course paramagnetic, if of repulsion, diamagnetic †.

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* We have $f=\frac{\partial W}{\partial x}$, $W=MH_y$, and $M=\mathcal{I}m$.

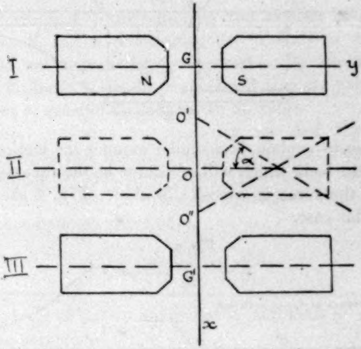
Hence $f=m\mathcal{I}\frac{\partial H_y}{\partial x}$.

† If the sense Ox is taken as positive and we employ the true formula for the force, $f=-\frac{\partial W}{\partial x}$ the negative sign for the force indicates attraction and the positive sign repulsion.

Suppose first that the body is paramagnetic.

When the magnet is situated so that the body O coincides with the centre G, the force will again be zero, as $\frac{\partial H_y}{\partial x}$ is zero at the centre (position II, fig. 2).

Fig. 2.

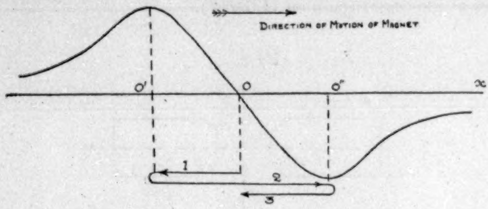


Hence the force f is zero when $x=\infty$ and $x=0$ and it passes through a *maximum* in the interval, which occurs at a certain point O′ where the product of the field H_y by its gradient $\frac{\partial H_y}{\partial x}$ is a maximum.

If the magnet is continually turned in the same sense of rotation, so that its centre G passes through the point O and arrives at G′ (position III, fig. 2), we observe that the force, which was nil at O, reaches a *minimum* or negative maximum at a point O″ symmetrical with O′, relatively to O, and becomes zero again after the magnet is withdrawn far from the body.

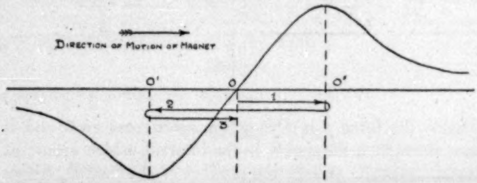
Hence the curve in fig. 3 shows the variation of the force f with the displacement x. The body is first displaced in the direction of the arrow marked 1 from O to O′, where it stops and moves back with the magnet along the arrow marked 2 to O″, where it again stops and afterwards returns to O, as shown by the arrow 3.

Fig. 3.



It is easy to explain in a similar manner the motions of a diamagnetic body which will of course be the reverse of the above, as the force is opposite in sign. Fig. 4 shows the curve in this case.

Fig. 4.



At every instant the action of the force is balanced by the torsion of the balance wire. If c is the moment of torsion per unit angle, and α the maximum deviation of the arm at O or O′ (fig. 2), the equilibrium condition is

$$fl = c\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

if l is the length of the balance arm.

If the deviation is measured by lamp and scale,

$$\alpha = \frac{D}{2L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

D being the deflexion of the spot upon the scale, and L the scale distance.

Hence from equations (3), (4), and (5)

$$K m H_y \frac{\partial H_y}{\partial x} l = c \frac{D}{2L}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which

$$K = \frac{cD}{2 m l L H_y \frac{\partial H_y}{\partial x}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This formula (7) thus permits the *absolute value* of the coefficient K to be determined for a body if the other quantities are accurately determined. By an analogous method P. Curie * has determined the absolute value of the coefficient of specific magnetization of water.

But the apparatus is especially adapted for relative measurements, and in this case it becomes extremely simple and practical.

If we have a body of unknown coefficient K and of mass m, we have from equation (6)

$$K m H_y \frac{\partial H_y}{\partial x} l = \frac{cD}{2L}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For a body of known coefficient K′ and mass m' occupying the same volume

$$K' m' H_y \frac{\partial H_y}{\partial x} l = \frac{cD'}{2L}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence by division

$$\frac{K}{K'} \cdot \frac{m}{m'} = \frac{D}{D'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (10) thus enables K to be determined.

It is evident that it is much more convenient to measure the displacements D and D′ of the spot corresponding to the angular deviations 2α and $2\alpha'$ between the two positions O′ and O″ of the body (fig. 2), for, without altering equation (10), the displacements of zero are eliminated and the accuracy of measurement doubled.

If the body is placed in a tube of glass or other substance, this tube being subjected like the body under test to the magnetic forces, an experiment must be made with the tube alone, and the effect due to the tube represented by a displacement of the spot D$''$ added or subtracted.

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* P. Curie, *Annales de Chimie et de Physique*, 7ᵉ Série, t. v. p. 344 (1895).

We thus have

$$\frac{K}{K'} \cdot \frac{m}{m'} = \frac{D \pm D''}{D' \pm D''} \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \quad (11)$$

Whence

$$\frac{K}{K'} = \frac{D \pm D''}{D' \pm D''} \cdot \frac{m'}{m} \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \quad (12)$$

This formula is not corrected for the effect due to the magnetization of the air ; the exact formula may be obtained as follows :

Let κ' and κ'' be the susceptibilities of the substance taken as standard, assumed to be paramagnetic, and of the air respectively. Let Δ' be the density of the comparison body, and A a constant of the apparatus. In reality the exact expression for the force when one deals with the standard body is

$$f' = (\kappa' - \kappa'')\frac{m'}{\Delta'}A = \left(K' - \frac{\kappa''}{\Delta'}\right)m'A, \quad . \ . \ . \quad (13)$$

since

$$K' = \frac{\kappa'}{\Delta'}.$$

When a measurement is made with a paramagnetic body of susceptibility κ and density Δ the true value of the force in this case is

$$f = (\kappa - \kappa'')\frac{m}{\Delta}A = \left(K - \frac{\kappa''}{\Delta}\right)mA. \quad . \ . \ . \quad (14)$$

Dividing (14) by (13) we have

$$\frac{f}{f'} = \frac{K - \dfrac{\kappa''}{\Delta}}{K' - \dfrac{\kappa''}{\Delta'}} \cdot \frac{m}{m'}$$

or

$$\frac{f}{f'}\,\frac{m'}{m} = \frac{K - \dfrac{\kappa''}{\Delta}}{K' - \dfrac{\kappa''}{\Delta'}} \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \quad (15)$$

If we put $r=\frac{f}{f'}\frac{m'}{m}$; this is the approximate ratio of the coefficients of specific magnetization K and K$'$ previously determined by the aid of formula (12) *.

From (15)

$$r=\frac{\mathrm{K}-\frac{\kappa''}{\Delta}}{\mathrm{K}'-\frac{\kappa''}{\Delta'}};$$

$$\mathrm{K}-\frac{\kappa''}{\Delta}=r\mathrm{K}'-r\frac{\kappa''}{\Delta'};$$

$$\frac{\mathrm{K}}{\mathrm{K}'}=\frac{\kappa''}{\mathrm{K}'\Delta}+r-r\frac{\kappa''}{\mathrm{K}'\Delta'};$$

$$\frac{\mathrm{K}}{\mathrm{K}'}=r\left[1+\frac{\kappa''}{\mathrm{K}'}\left(\frac{1}{\Delta r}-\frac{1}{\Delta'}\right)\right], \quad . \quad . \quad . \quad (16)$$

in which

$$r=\frac{\mathrm{D}\pm\mathrm{D}''}{\mathrm{D}'\pm\mathrm{D}''}\cdot\frac{m'}{m}.$$

When the comparison body is water

$$\mathrm{K}' \text{ exactly } = -0{\cdot}79\times10^{-6} \text{ (P. Curie).}$$

As for air the susceptibility $\kappa''=0{\cdot}0322\times10^{-6}$.

If r is *positive* we have for a paramagnetic body

$$\frac{\mathrm{K}}{\mathrm{K}'}=-r\left[1+0{\cdot}041\left(\frac{1}{r\Delta}+1\right)\right],$$

and for a diamagnetic body

$$\frac{\mathrm{K}}{\mathrm{K}'}=r\left[1-0{\cdot}041\left(\frac{1}{r\Delta}-1\right)\right].$$

Hence the apparatus lends itself very readily to relative determinations, and if we take water as the standard body

---

* We have in fact, from what has been proved above,

$$fl=\mathrm{K}m\mathrm{A}=\mathrm{B}(\mathrm{D}\pm\mathrm{D}''),$$
$$f'l=\mathrm{K}'m'\mathrm{A}=\mathrm{B}(\mathrm{D}'\pm\mathrm{D}'');$$

A and B as well as l being constants; whence

$$\frac{f}{f'}=\frac{\mathrm{K}m}{\mathrm{K}'m'}\quad\frac{\mathrm{D}\pm\mathrm{D}''}{\mathrm{D}'\pm\mathrm{D}''};\qquad\frac{fm'}{f'm}=\frac{\mathrm{K}}{\mathrm{K}'}=\frac{\mathrm{D}\pm\mathrm{D}''}{\mathrm{D}'\pm\mathrm{D}''}\cdot\frac{m'}{m}.$$

($K' = -0{\cdot}79 \times 10^{-6}$) it enables us to obtain an absolute value of the coefficient of specific magnetization of a body.

But the comparison substance may be a liquid or solid other than water, and such that its coefficient of magnetization is of the same order of magnitude as that to be determined. A choice can be made from the tables of magnetic constants. It is convenient if the masses of the bodies compared correspond to the same volume; on this account the tube containing the substance is always filled to a given mark. When the substances tested are very strongly magnetic, a smaller length may be employed, but in that case it is absolutely necessary that the comparison substance should have the same length.

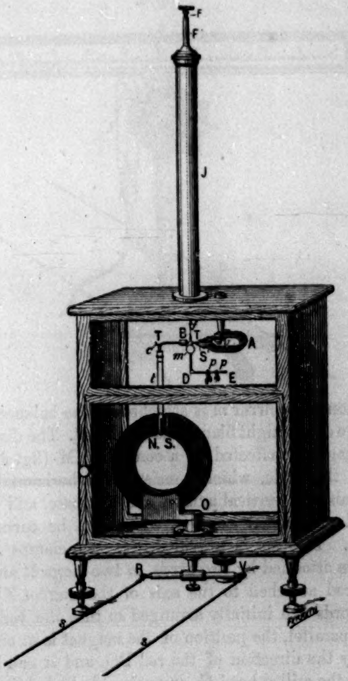
The apparatus could be made still more sensitive with the aid of an electromagnet. On the other hand, with a torsion wire of larger diameter, the magnetic properties of iron or other ferro-magnetic metals and alloys could be determined. It would doubtless suffice to use an extremely fine wire of the metal, and to employ the method of experiment and the corrections already proposed by P. Curie in a work on the magnetic properties of bodies (*loc. cit.* p. 347, above).

Description of the Instrument, Magnetically Damped Form.

The arm of the torsion-balance is formed by a rod TT (fig. 5), which carries from one of its two extremities a hook *c*, from which the glass tube *t*, which contains the body, can be suspended. To this end the glass tube is closed by a cork carrying a metallic ring which hangs on the hook *c*. The tube *t* is therefore in a vertical position. On the other end of the arm TT is fixed a copper sector, S′, which moves between the poles of a fixed damping magnet A. To vary the sensitiveness the torsion-wire can be changed, and the damping varied by the position of the copper sector in the field.

On the horizontal arm TT is fixed a vertical copper strip which carries a hook to which the torsion-wire *f* ($\frac{1}{10}$ mm. platinum) is attached. Below, this strip is turned at right angles as shown by the portion DE parallel to TT. Balance is obtained partly by the aid of a small copper cylinder B, sliding on the rod TT, when the tube is empty, and partly by copper or aluminium riders, *pp*, to compensate for the weight

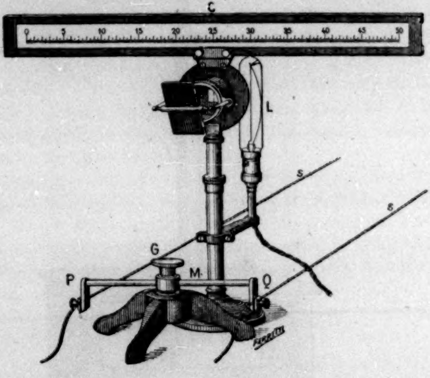
Fig. 5.



of the solid or liquid material which fills the tube to the level marked.

The displacement of the magnet N.S. is obtained by a movement of rotation around an axis O; this movement is smoother than the sliding motion of our earlier model. The movements of the torsion-balance are followed by the observer upon a divided scale C. (fig. 6). For this purpose a 2-metre

Fig. 6.



radius concave mirror *m* is attached to the balance-arm, and the image of a straight filament is employed. The displacement of the magnet is effected by a controller M (fig. 6) fixed in front of the scale, which consists of a horizontal rod PQ turning about a vertical axis in a heavy base, and furnished with a large milled head G, which can be turned by the observer. This rod is provided with two clamps P and Q, which are attached by two cords to two rings R and V on a similar rod attached to the axis of the magnet NS (fig. 5). If the cords are initially arranged so that the rods PQ and RV are parallel, the position of the magnet is at once determined by the direction of the rod PQ, and it can be turned either by the milled head G, or preferably by holding the two cords *ss* like reins.

The advantage of employing a rectilinear filament lamp (such as is made by the Pintsch Gluhlampenfabrik in Berlin) is considerable in practice, as the spot is always visible on

Substance.	Density, Δ.	Coefficient of Specific Magnetization, K.
Water	1·0	−0·79 × 10^{-6}
Alcohol	0 79	−0·78
Amyl Alcohol ..	0·815	−0·929
Benzene	0·85	−0·962
Luxor Oil	0·8	−0·594
Methyl Ether ..	0·725	−0·785
Aluminium	2·67	1·19 × 10^{-6}
„	..	1·79
„	..	2·03
„	..	2·99
Brass	8·1	−0·026 × 10^{-6}
„	..	3·95
„	..	0·3
„	..	0·09
„	..	0·083
„	..	2·02
„	..	1·94
„	..	0·232
„	..	3·09
„	..	4·05
„	..	2·45
Copper	8·9	0·707 × 10^{-6}
„	..	0·455
„	..	1·06
„	..	0·68
„	..	0·54
„	..	0·36
„	..	0·83
„	..	0·537
„	..	−0·097
Tin	7·3	0·151 × 10^{-6}
Zinc	6 9	0·043 × 10^{-6}
„	..	0·212
„ (Pure)	..	0·224

Substance.	Density, Δ.	K.	Specific Resistance, Microhms per cm. cube at 0° C. ρ.	Temperature Coefficient % at 17° C.	Thermo E.M.F. to Cu Microvolts per ° C.
Nickelin, I	9·09	4·88 × 10^{-6}	43·13	0·00245	37·5
Nickelin, 1	8·77	3·75	40·62	0·021	20·0
Nickelin, 5	8·79	2·11	39·29	0·0187	19·06
Nickelin, 13	8·82	5·76	39·73	0·021	20·75
Extra Prima	8·8	2·1	29·35	0·028	16·55
Platinoid	8·83	1·75	33·62	0·0224	23·52
„	..	1·61			
„	..	1·63			
Constantan Vogel	8·99	9·67	47·06	−0·00122	37·5
„	..	10·25			
„	..	9·72			
Constantan, 12	8·97	12·37	48·3	0·001437	35·75
1A 1A	8·97	9·23	50·7	−0·0029	39·12
German Silver, 2 A	9·06	6·77	42·89	0·00385	35·75
New Metal	8·97	6·87	51·10	−0·0038	35·8
„	..	6·77			
Eureka	8·96	..	47·4	0 0048	40·6
Manganin, 14	8·61	28·05	36·62	0·00175	1·4
Manganin Isabelenhutte ..	8·6	28·7	39·14	0·00176	1·44
Rheotan	8·61	..	47·55	0·024	13·8
Superior	8·24	..	81·024	0·1148	7 44

Effect of Volume on Value of K.

Nickelin, I.

Mass, *m*.	K.
0·198	2·04 × 10^{-6}
1·985	1·99
1·784	2·16
1·583	2·23
1·385	2·03
1·187	2·11
0·989	2·13
0·789	2·14
0·592	2·18
0·393	2·18
0·192	2·41

Extra Prima.

Mass, *m*.	K.
0·21	2·1 × 10^{-6}
0·434	1·52
0·635	1·6
0·836	1·52
1·033	1·66
1·233	1·29
1·435	1·38
1·637	1·29
1·835	1·28
2·033	1·19
2·237	1·23
2·437	1·13
2·634	1·07
2·834	1·09

Effect of Length on Value of K.

Copper.

Length.	K.
5 cm.	0·404 × 10^{-6}
4·5	0·446
4·0	0·516
3·5	0·642
3·0	0·828
2·5	0·71
2·0	0·895
1·5	1·093
1·0	1·46
0·5	2·37

Effect of Acid Treatment.

Copper.

Mass, *m*.	K.
2·545 gr.	1·16 × 10^{-6}
After Treatment with Hot Conc. HCl:—	
2·542	0·765
Further Treatment with HCl:—	
2·540	0·774

the scale even if the torsion-arm is not perfectly balanced horizontally.

Appendix.

By the courtesy of M. Cheneveau and the Cambridge Scientific Instrument Co., we have been able to set up and test the performance of one of these balances in the laboratories of the Northampton Institute.

The instrument was of the magnetically damped type described above, and was used with a lamp and scale at some two metres distant from the mirror.

No attempt was made to obtain absolute values of the coefficients of specific magnetization; but relative determinations only were made, using as a standard substance distilled water, and assuming for it the value

$$K' = -0{\cdot}79 \times 10^{-6},$$

as found by M. Curie and given above.

The results obtained are set out in the accompanying table, and show some interesting figures.

The first portion of the table contains figures for a few materials selected quite at random in order to test the behaviour of the balance. Of these the first five are chemically pure liquids, and all exhibit diamagnetic qualities.

The four samples of aluminium are all magnetic, and we have not been able to get a sample which is less magnetic than the first of these. The eleven samples of brass indicate the range over which the magnetic qualities vary, while some of the samples examined, but not here tabulated, have been too magnetic to be used in the apparatus without changing the suspension.

It is interesting to note that during some inductance tests in progress in the laboratory, circumstances compelled us to use brass conductors, and the results obtained were higher than those given by calculations based on theory. This could only be accounted for by assuming a permeability for the brass greater than unity, and although the magnetism in the metal was not detectable by a fairly sensitive magnetometer, it was readily shown when samples of the conductors were tested in the balance.

Nine samples of copper are tabulated, and one only is diamagnetic, again emphasizing the difficulty of obtaining really non-magnetic conductors.

The tin sample was chemically pure and is also magnetic. Of the three samples of zinc the first two are ordinary commercial sheet metal, and the third one is a rod of chemically pure metal for use in standard cells.

The next portion of the table is devoted to an examination of the series of resistance alloys, whose electrical properties were investigated by the writer some time ago and published by Dr. Drysdale in connexion with his paper upon the Comparison of Standard Resistances, before the British Association at Leicester, 1907.

The first point which strikes one in connexion with these is the fact that every alloy is magnetic, but that those alloys containing relatively a large proportion of the magnetic metal nickel, *i.e.* Nickelines, are among the least so. The first sample is very different from the other three, and we have reason to believe that the second and fourth samples are of the same manufacture although obtained from different firms.

The samples of Platinoid fall into this group with properties very comparable with the Nickelines, and the material sold under the trade name of Extra Prima probably also belongs to this group.

The Constantans come next in order, being more magnetic than the Nickelines, and it is not difficult to see that the material catalogued as 1A1A belongs to this group of alloys, together with the alloy supplied to us under the title of New Metal, while the sample German Silver 2A, but for its high density and positive temperature coefficient, would also be classed among the Constantans.

Eureka is a resistance material which is often classed as a Constantan, but the balance easily disproves this, as it shows it to be so magnetic that observations would have had to be made with a stiffer suspension than we were using, and this, coupled with its positive temperature-coefficient, rule it out of the group.

Superior and Rheotan behave like steels in the apparatus, and probably contain considerable quantities of iron in their composition.

The magnetic behaviour of Manganin is remarkable, it being, with the exception of the last two alloys and Eureka, the most magnetic of all the materials examined. This is of particular interest in view of its almost universal adoption for the construction of accurate resistances, but it is not altogether unexpected when we remember that the remarkable Heusler alloys have a Copper-Aluminium Manganese composition.

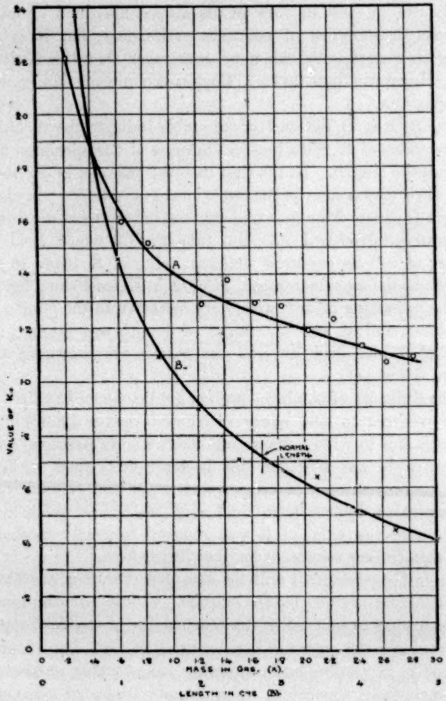
The figures in the rest of the table indicate the variation in the value of K, with increase in mass of the specimen and increase in length. In the first case the sample, in the form of a bare wire 1 mm. in diameter, was cut up into standard length (approx. 2·8 cms.), and successive lengths of measured mass were introduced into the tube until it would hold no more; it will be observed that the value of K tends to diminish as the mass increases. (Curve A, Extra Prima, fig. 7.)

The apparatus is, of course, very sensitive to the length of specimen, and an extreme range of length was taken in a wire of 1 mm. diameter, and the results are shown in the curve B (copper).

The difficulty of obtaining non-magnetic copper has already been referred to, and every instrument-maker knows how difficult it is to wind a really non-magnetic D'Arsonval coil. In order to see how far this is dependent upon surface conditions a sample of copper was taken and treated with concentrated hydrochloric acid and re-tested, and it was found that the value of K was reduced to nearly one-half, and that further treatment had but little effect.

From these results it will be seen that there is a considerable field of usefulness for the balance. Firstly, for examining the materials to be used in the construction of standard apparatus where the permeability becomes of first importance. Secondly, in grading and examining alloys whose properties are materially dependent upon minute traces of alloying constituents which are so difficult to estimate by chemical means, and which play such an important part in their electrical behaviour. Thirdly, it could well be used to maintain the standard of purity in the commercial production of such a metal as Aluminium, whose chief impurity is iron, the last traces of which are so difficult to remove.

Fig. 7.



In conclusion the writers' thanks are due to Mr. A. F. Burgess, B.Sc., for his help in the experiments and calculations.

ABSTRACT.

This balance is intended for the determination of the coefficient of specific magnetization, susceptibility, and permeability of feebly paramagnetic and diamagnetic bodies. The body under investigation is suspended from one arm of a torsion balance, which measures the force exerted on the body when it is placed in the non-uniform field of a permanent magnet. The torsion balance is formed by a horizontal rod suspended by a long fine platinum wire and carrying at one end a hook from which the substance under investigation can be suspended in a small enclosing glass tube. On the other end of the torsion arm a copper sector is fixed which moves between the poles of an auxiliary magnet and thus provides efficient damping. A second branch arm is also provided upon which may be placed suitable counterweights to balance the specimen. The suspension carries a mirror, and the movements are read on a translucent scale in the ordinary way. The magnetic field is that of a large circular permanent magnet mounted in such a way that the vertical gap may be made to describe a semicircle of radius equal to the length of the torsion arm, and the movements of this magnet are controlled by the observer at the screen by means of cords. An empty containing tube is first mounted on the balance, and the maximum deviation on both sides of the zero is obtained by bringing up the magnet towards each side of the specimen. The tube is now filled to a given mark with a known mass of pure distilled water and the variations observed as before. The water is now replaced by a known mass of the substance to be examined occupying the same volume or the same vertical height, and again the deviations are observed. The paper indicates how the value of K may be calculated from these observations. Experiments have been performed with various metals and alloys, and the results are tabulated and discussed in the paper.

DISCUSSION.

Mr. A. CAMPBELL said that for certain experiments, and in the construction of certain instruments it was essential to know that the materials used were non-magnetic; and he thought the Author's instrument would be useful. It was very difficult to obtain non-magnetic brass.

Mr. W. DUDDELL asked if the specimens used were specially cleaned, as the oxides and sulphides of many metals were magnetic. It would also be necessary to keep them clean to get trustworthy results.

Mr. D. OWEN expressed a doubt as to the accuracy of the results attainable by this apparatus used as described by the Authors in the paper. Theory requires that the specimens of the standard substance and that under test should be exactly equal in size and shape, and similarly situated in the variable magnetic field when the observation of maximum force is taken. In many ways this condition may be departed from in the experiments described. The size of the test-tube used to hold the specimen is a considerable fraction of the interpolar space, and, excepting in the case of liquids, it would be difficult to ensure identity of form of the samples in the correlated pair of experiments. Absence of coincidence between the axis of rotation of the magnet and the axis of the torsion wire would cause an error of position varying with the deflection of the balance-arm. Again, the magnetic pull parallel to the balance-arm will cause movement of the specimen to an extent depending on its susceptibility and its density, though the effect is likely to be very small. These causes may account for the Authors' observations of apparent dependence of K on the mass and length of specimen.

The Authors were also asked whether they had any experimental data as to the independence of K with respect to temperature.

Dr. Russell regretted that the Author had not explained more fully how the magnetization of the specimen under test was maintained uniform. Any lack of uniformity would seriously affect the accuracy of the results. He thought that the great discrepancies found by Mr. Jolley in the apparent value of the coefficient of the specific magnetization of copper for specimens of different lengths must be due to an uneven distribution of the magnetic field. When the specimens were very feebly magnetic, he considered that it was legitimate to make the assumption that the intensity of the magnetization was proportional to the field. He did not see, however, how the balance could be employed to test strongly magnetic substances. He suggested that the specific magnetization of water might vary largely with temperature. It was interesting to find that the discrepancies between the calculated and the observed values of the inductance of a certain brass conductor had been traced to the magnetic qualities of the brass. In his opinion errors in the computed values of inductance standards, due to the permeability of the conducting material differing from unity by a very small fraction, would in general be negligible.

The Secretary read a letter from Dr. C. V. Drysdale in which he stated that Mr. Jolley's experiments with the magnetic balance proved its value, both scientifically and commercially, as a means of rapidly estimating the composition and purity of alloys and other materials. The instrument enabled the magnetic qualities of materials to be obtained as easily as the specific gravity measurements which are usually employed as an aid to practical analysis, and it formed a kind of magnetic hydrometer which gave the specific magnetization instead of the density of the substance as compared with water. The displacement principle of Poisson and the derivation of the force as the slope of potential were both of great importance, and an apparatus such as the Author's which showed the practical application of these principles was welcome.

Mr. C. W. S. Crawley drew attention to the fact that if the brass rods were cut with steel shears they would be rendered magnetic. Brass tubes prepared by drawing through steel dies were generally magnetic. He asked if the variation of the value of K for copper was due to impurities.

Mr. R. S. **Whipple** asked if the Author had performed any experiments on wood or ebonite, and, if so, with what results.

Mr. **Jolley**, in reply, said he could fully endorse Mr. Campbell's remarks concerning the difficulty of obtaining non-magnetic brass which seemed to be quite rare. In reply to Mr. Duddell's point concerning the cleaning of the specimens, a fair amount of care was taken to keep the specimens clean; they were handled with ivory forceps, and in most cases they were bright new samples of rod and wire, which to the eye certainly did not seem in any way oxidized; and if any such oxide or other salt were present, unless it were very highly magnetic it would not be sufficient in quantity to account for the high values of K generally obtained.

Mr. Owen raised a point in connexion with the theory, which would be of importance if the maximum deflexion were obtained with the specimen in the interpolar gap, but, as a matter of fact, the maximum deflexion as indicated in the paper is always obtained when the specimen is considerably out of the gap of the magnet, and consequently the position of the specimen does not become so important. An excellent datum is, however, provided by the damping sector, which has to be accurately between the poles of its magnet for perfectly free rotation, and when this is so the tube is rightly centred relatively to the gap of the deflecting magnet. No experiments have been made either as to side pull, which however in feebly magnetic bodies must be very slight indeed, or as to temperature.

With regard to the uniformity of magnetization of the specimen raised by Dr. Russell, he believed that some experiments on this had been made by MM. Curie and Cheneveau, but the point would have to be referred to M. Cheneveau before anything definite could be said. With regard to Mr. Crawley's point as to the cutting of the specimens and the presence of iron from the tools employed in manufacture, they had no special method of cutting the lengths and had to resort to the usual workshop processes. The experiments with acid treatment show that although in some part the magnetic quality is a surface effect, this is by no means accountable for all, and the amount of magnetic material left on the surface from tooling was far too small to account for the values of K found. With regard to copper, the value of K certainly was due to impurity, but it is considered that the magnetic balance gives indications of the presence of such impurity with far greater reliability than any chemical test for such minute quantities.

In reply to Mr. Whipple, both wood and ebonite had been tested but not tabulated. In both these cases the results were variable, and it was practically impossible to free the material from the effects of tooling; in any case the results always showed the specimens to be very magnetic.

XXV. *On an Oscillation Detector actuated solely by Resistance-temperature Variations.* *By* W. H. Eccles, *D.Sc.*[*]

In a recent communication to the Physical Society[†] the properties of a type of iron-oxide coherer were discussed. The paper described experiments on coherers made by dipping a slightly oxidised iron wire into clean mercury, or by pressing a fine iron wire against a thinly oxidized iron plate, and the results were discussed mathematically. It was shown that in the case of the iron point and oxidized plate the whole of the experiments could be explained qualitatively on the assumption that the only electrical phenomena at play were the Joule effect and the resistance-temperature changes in the small mass of oxide between the metal electrodes. The hypothesis that was put forward must be summarized here. Let ρ be the resistance of that part of the detector where the current flow is so constricted that the Joule effect produces rise of temperature, and r the resistance of the remainder of the circuit. The resistance ρ is usually localized at the contact of the conductors that form the detector, and it varies with the temperature of the minute mass of matter at the contact; let α be the coefficient of decrease of resistance with rise of temperature. The resistance r includes that of the bulk of the substances forming the detector, the leads and the telephone; it is supposed constant. Let c be the current when the electromotive force applied to the detector is ϵ. Assume that the rate of loss of heat from the warmed contact is $m\theta$, where θ is the temperature of the contact above that of the surroundings. Then it was shown that in the steady state

$$\epsilon=\left(\frac{\rho_0}{1+q\alpha c^2}+r\right)c, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ρ_0 is the resistance of the contact when cold and q is a constant. If the curve $\epsilon=\rho_0 c/(1+q\alpha c^2)$ be plotted with ϵ as abscissæ and c as ordinates, it is seen to rise from the origin of coordinates with an increasing gradient till at a definite value of e it becomes vertical. Then as c increases, the curve bends towards the axis of c and approaches it asymptotically. Along this latter part of the curve, increasing current is associated with decreasing electromotive force—an unstable state of affairs. Any conductor possessing a negative temperature-coefficient of resistance exhibits these properties. In such conductors an increase of current produces, in accordance with Joule's law, an increase of temperature, and consequently a diminution of the resistance. The curve shows that after a certain stage is passed the diminution of the resistance which accompanies increasing current is great enough to lead to a catastrophe. Stability can, however, be obtained by introducing into the circuit of the variable resistance ρ a sufficiently large constant resistance r. The phenomenon resembles that of the electric arc. The unstable portion of the above curve corresponds, in fact, to the "falling characteristic" of the arc.

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[*] Read May 27, 1910.

[†] W. H. Eccles, "On Coherers," Phil. Mag. June 1910; Proc. Phys. Soc. vol. xxii. 1910.

If the resistance r is large the e, c curve has a positive gradient throughout, and the gradient has a maximum at some value of e. Near this point the contact is found to work best as a detector of oscillations. The hypothesis put forward by the author supposes that a train of oscillations, by yielding its energy as heat to the contact, raises the temperature there and disturbs the existing equilibrium of current and voltage. The dissipation of a train of oscillations is accomplished in a time of the order of a fifty-thousandth of a second; the ensuing fluctuation of the current causes the sound in the telephone. In the paper cited, the energy w given to the telephone circuit was shown to be connected with the energy W delivered by the train of waves by the equation

$$w=m(\mathrm{W}-a), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

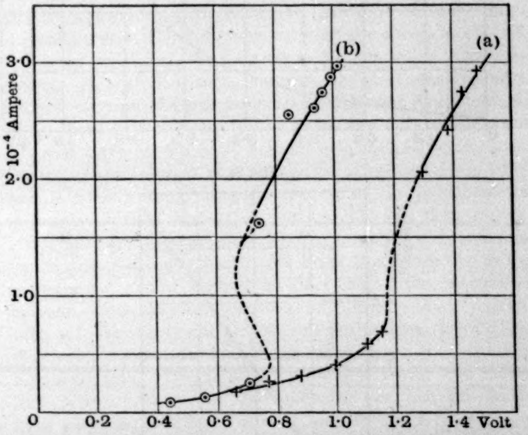
where m and a are constants for any fixed value of the current. This theoretical conclusion agreed well with the experiments on an iron oxide coherer. The object of the present paper is to show that the above hypothesis holds good for a very different type of detector.

All the well-known forms of "contact" detector consist of a contact between two substances that stand well apart in the thermoelectric series; and the thermoelectric force plays a very important part in their operation. They are usually classed as "rectifying detectors" to distinguish them from such detectors as the coherers. In seeking a "rectifying detector" that would illustrate the above hypothesis, substances that stand far apart in the thermoelectric series must be avoided, for such substances would introduce Peltier effects that would tend to mask the resistance-temperature phenomena we wish to isolate. Even in the iron : iron-oxide : iron coherer it is possible that thermoelectric forces arise owing to unequal heating of the two iron to iron-oxide junctions. It occurred to me, therefore, to construct a detector out of one substance only. Accordingly a search was made for a substance possessing high, but not too high, resistivity, with large negative temperature-coefficient. Several such substances were found among the native crystalline oxides and sulphides. Most of these are ælotropic, and must on that account be avoided. Fortunately galena, the native sulphide of lead, has fairly high resistivity, a very large negative resistance coefficient, and crystallizes in the cubic system. A galena-galena detector was therefore constructed. Two pieces of galena cut from the same crystal were embedded in solder, mounted in a clamp, and brought into gentle contact. It was put into a wireless telegraph receiving set and was found to yield excellent signals when a current of proper magnitude was passed through it.

A few preliminary experiments showed that it was not easy to find two pieces of galena, which when placed together and submitted to positive and negative voltages in turn, exhibited perfect symmetry. The asymmetry is always small, and is probably due to the rise of thermoelectric forces at the contact when it becomes heated by the steady current, and these forces are probably caused by slight variations in the chemical composition of the galena from point to point in the crystal. When a contact that gives symmetrical voltage-current curves is found, it constitutes a detector that operates only when a current is passing through it. The efficiency of the detector is the same for each direction of the current, and appears to be practically independent of the shape of the galena surfaces at the contact.

The experiments now to be described were made on selected galena-galena combinations in precisely the same way as the earlier experiments on the iron-oxide detectors. The diagram of the apparatus employed is shown in fig. 1 of the previous paper (Phil. Mag. June 1910, p. 872, & Proc. Phys. Soc. vol. xxii. p. 293). The mutual inductance between L and L′ was 2500 cm. in obtaining the results described below.

Fig. 1.



The detector was placed at D and submitted to an electromotive force from P_2, which was varied in steps from zero to about one volt positive or negative. At each step an observation was made of the steady current through the detector, and of the intensity of the sound produced in the telephone by electrical oscillations of amplitude fixed by the position of the jockey on P_1. The steady current observation was made by means of a shunted galvanometer kept connected in series with the telephone. The intensity of the sound was measured by balancing the sound from the detector against that from the interrupter and the potential-divider P_3. Thus were obtained the curves of figs. 1 and 2. Fig. 3 was obtained by applying oscillations of various amplitudes to the detector while it is traversed by a suitable steady current, and measuring the intensities of the resulting sounds; and therefore expresses the relation between the energy given to the detector and that appearing in the telephone circuit.

Fig. 2.

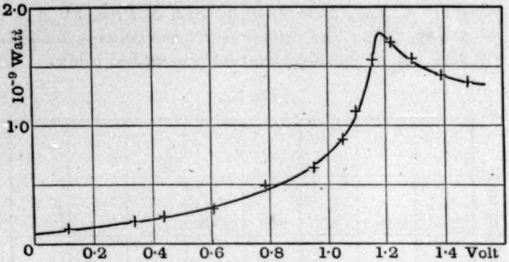
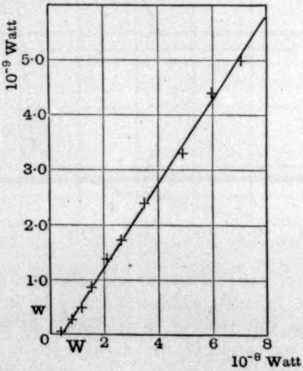


Fig. 3.



The curves given here are selected out of a number as representative ones. The absolute values of the energy quantities involved in the results given in this paper were determined in the same way as in the former paper, and are therefore subject to the same errors; the errors in the *absolute* values of the energy delivered to the telephone circuit may be large.

The most difficult observations to make were the steady current ones. The current-voltage curves obtained with increasing currents are very different from those got with decreasing currents unless certain precautions are observed. Usually a decreasing-current curve lies above an increasing-current curve. The difference between the curves was found to depend greatly on the time allowed for a set of measurements. If, with decreasing currents, the electromotive force be held constant, the galvanometer reading slowly diminishes, till in about half an hour it has fallen as a rule to the value it would have at the same voltage with increasing currents. On the other hand, if the voltage be raised suddenly from zero to a fairly large value, the reading of the galvanometer increases as time elapses, at first quickly and then slowly, rising perhaps fifty per cent. in five minutes, and asymptotically approaching a limit. All these things are due to the temperature at the contact lagging behind the changes of current. This may be understood as follows:—At any stated value of the steady current the mass of galena surrounding the heated contact must be cooler when the current is being increased than when it is being decreased—for the reason that in the former case the conductor has just previously been carrying smaller currents, in the latter case larger currents, than the particular current considered. The rate of loss of heat from the heated matter at the contact to its surroundings is thus greater on the rising curve than on the falling curve, and, in consequence, at corresponding points on the two curves the resistance is higher in the former than in the latter case. This temperature lag may be expected to be more pronounced in detectors made wholly of substances of low thermal conductivity than in detectors consisting of a very thin film of oxide between metal electrodes. The phenomenon is very prominent in this new galena detector though it escaped observation in the iron-oxide detectors, and therefore the former may be expected to work well as a rectifier of alternating currents of moderate frequency.

Once the slow movements of heat through the galena had been observed the difficulties were overcome by allowing time for thermal equilibrium to be attained before the final galvanometer reading was taken. At places near the point of inflexion of the steady current curve the time necessary may be several minutes.

Each setting of the two pieces of galena gives a different contact resistance and therefore a different curve. Since very slight accidental vibrations can cause relative motion of the pieces of galena, it is almost impossible to perform a series of measurements yielding sufficient data for all the curves of figs. 1, 2, 3, with full confidence that the contact has not varied.

Curve *a* of fig. 1 follows the equation.

$$\epsilon=\left(\frac{3\cdot 1\times 10^4}{1+2\cdot 95\times 10^8c^2}+4\cdot 0\times 10^3\right)c \quad . \quad . \quad (3)$$

with fair accuracy; curve *b* obtained with another setting of the crystals has the equation

$$\epsilon=\left(\frac{4\cdot 0\times 10^4}{1+8\cdot 3\times 10^8c^2}+2\cdot 5\times 10^3\right)c. \quad . \quad . \quad . \quad (4)$$

The curve of fig. 2 was obtained with the same setting of the crystal that gave curve *a* of fig. 1. It was shown in the previous paper that the ordinates of this curve should be proportional to *m* the coefficient of W in equation (2) above, and that *m* contained the gradient of the steady current curve as its principal factor. A comparison of figs. 1 and 2 confirm the deduction. The line of fig. 3 has the equation

$$w=0\cdot 077\ (W-0\cdot 4\times 10^8),$$

where W is the power in watts given to the detector in the form of oscillations, and *w* the power transformed by the detector and passed to the telephone circuit.

The galena detector here investigated proves very good in practical wireless telegraphy. It has shown itself to be better than any coherer * known to me. This superiority is, according to the hypothesis here advocated, to be ascribed chiefly to the large negative temperature-coefficient of resistivity of galena. By direct measurement of a cube of galena clamped between pieces of tinfoil the resistance was found to fall from 0·33 ohm at 12° C. to 0·10 ohm at 99° C. —a negative coefficient of 0·0079 per degree centigrade. Pyrites has a coefficient about 0·006. Iron oxide has a coefficient somewhat lower than this last.

The cost of a portion of the apparatus used in these experiments was defrayed out of a grant from funds at the disposal of the Royal Society.

ABSTRACT.

The experiments described in the present paper are offered as additional support for the Author's hypothesis of the mode of action of certain types of electrical oscillation detectors. This hypothesis suggests that in detectors constituted of a loose contact, the energy of the oscillatory current through the contact is transformed into heat at the contact and warms the matter there sufficiently to change its electrical resistance, and, consequently, the steady current through the indicating instrument. The principal deductions from this hypothesis were worked out and were illustrated by experiments on iron-oxide coherers in a paper read before the Physical Society on the 11th of March last. The present experiments are on a detector of the so-called "crystal rectifier" type, from which, however, the possibility of thermoelectric effects have been eliminated. This detector consists merely of a loose contact between two pieces of galena—a substance which according to the Author's theory ought, by virtue of its large negative coefficient of change of resistance with temperature, to be a very efficient detector of electrical vibrations. The experimental curves obtained from a galena-galena detector are: the steady current curve, the sensitiveness curve, and the power curve. The first has steadily applied E.M.F. as abscissæ, and current through the detector circuit as ordinates. It proves to be a curve which in general rises slowly at first, then quickly, and then slowly again; but if the circuit be arranged to have but little resistance other than that at the contact, the curve may possess a negative gradient at the point of inflexion. The chief difficulty met in obtaining these steady-current curves arose from the slow movements of heat through the mass of the crystals—which have, of course, small thermal conductivity. This difficulty was overcome, and the true character of the curves brought to light, by allowing a proper time-interval for thermal equilibrium

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* But not so good as certain "thermoelectric" detectors.

to be established before each galvanometer-reading was taken. The sensitiveness curve has the E.M.F. applied to the detector as abscissæ, and the power passed to the indicating instrument as ordinates: the intensity of the electrical vibrations being of fixed amount. It is a curve which rises to a maximum near the point of greatest slope of the former curve, and thereafter descends slowly. The power curve has the power supplied to the detector in the form of electrical oscillations as abscissæ, with the power passed by the detector to the indicating instrument as ordinates, the steady E.M.F. applied to the detector being the best value. This curve is a straight line. The properties of this "crystal rectifier" are therefore just such as are logically deducible from the fundamental fact that the contact possesses a negative resistance-temperature coefficient.

DISCUSSION.

Mr. W. **DUDDELL** said the Author was making progress with his thermal theory of coherers. He pointed out that in the experiments described considerable time was taken in determining the part of the steady current curve with negative characteristic, whereas in practice the changes were very rapid.

Prof. C. H. **LEES** remarked that Dr. Eccles had introduced simplicity into a very difficult subject. He agreed with the remarks of Mr. Duddell, and suggested using very minute points and surfaces so that less time would be occupied in securing thermal equilibrium. The working conditions would thus be more nearly reproduced.

The **AUTHOR**, in his reply to the remarks of Mr. Duddell and Prof. Lees, explained that in order to get a correct notion of the character of the steady current curves, the curves should be drawn either very quickly or very slowly, that is to say, either adiabatically or isothermally. The former is really the curve required for strict quantitative correlation with the experimental facts of the behaviour of the detector towards high-frequency electrical oscillations. The isothermal curves actually obtained are the same *in character* as the adiabatic ones, to a sufficiently close approximation. If the masses of the crystals were reduced, as Prof. Lees suggested, in order to reduce the heat-losses from the contact, then the metal leads to which the crystals are soldered might be brought near enough to the contact to make the heat-losses larger, not smaller. The chief advantage of making the crystals smaller accrues from the reduction of that idle portion of the electrical resistance of the detector, which undergoes no temperature change of the kind desired. The Author practically accomplishes this already by copper-plating the crystals all over, soldering into place, and then baring the small area of crystal required for the contact.

XXVI. *The Limitations of the Weston Cell as a Standard of Electromotive Force. By* S. W. J. SMITH, *M.A., D.Sc., Lecturer on Physics, Imperial College of Science and Technology.*[*]

§1. THIS paper contains an attempt to explain Mr. F. E. Smith's recent experiments on the cadmium amalgams of the Weston cells [†] in terms of the theory of solutions.

The manner in which, according to this theory, the amalgams crystallize is indicated, and it is shown why, with this mode of crystallization, the effect of the slowness of diffusion is so pronounced, and also why sudden cooling to a temperature below the freezing point of mercury must produce a comparatively uniform alloy.

The cause of the most obvious differences between the behaviour of the "chilled" and of the "slowly cooled" amalgams is then at once apparent.

But there are certain much less obvious differences of which, in view of the importance of the Weston cell as a standard (if for no other reason), it is desirable to know the cause.

An all-fluid amalgam, of course, yields the same E.M.F. whether previously chilled or cooled slowly. An all-solid alloy, on the other hand, always gives a higher E.M.F. when solidified by chilling.

The E.M.F. of the chilled amalgam thus in general equals or exceeds that of the slowly cooled amalgam.

But in the range of the two-phase alloys, over which the E.M.F. varies comparatively little with the total percentage of Cd, the opposite is true. The E.M.F. of the slowly-cooled amalgam now either equals or exceeds that of the chilled amalgam.

The excess never amounts to more than a few hundred-thousandths of a volt; but is important in measurements of the highest precision and requires explanation.

It is shown that this phenomenon may be due to electrolytic surface effects arising from the lightness and want of uniformity of composition of the solid grains in the slowly cooled amalgams.

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[*] Read May 27, 1910.

[†] Proc. Phys. Soc. vol. xxii. 1910, pp. 11–40.

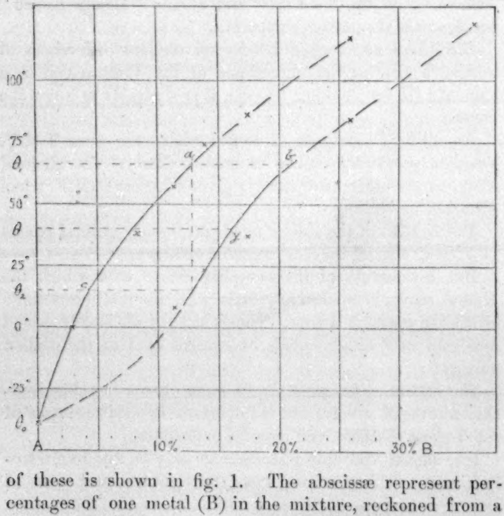
Instances of similar effects, liable to escape notice, are to be found amongst the data for the all-solid alloys.

Finally, the question as to whether there is any range over which the E.M.F. is absolutely independent of the percentage of cadmium is discussed.

Theory and experiment alike suggest that the E.M.F. must rise as the percentage increases; but the variation frequently does not amount to more than a few millionths of a volt for one per cent. variation of the cadmium content.

§ 2. *A mode of crystallization of binary alloys.*—The various ways in which fluid mixtures of two metals can freeze have been carefully studied within recent years. One of these is shown in fig. 1. The abscissæ represent percentages of one metal (B) in the mixture, reckoned from a zero at which the other metal (A) alone is present. The

Fig. 1.



ordinates represent temperatures. The melting point of A is θ_0. The "freezing point curve" $\theta_0 xa$ gives the temperatures at which different alloys begin to solidify, and shows that the freezing point rises continuously as the percentage of B in the mixture increases. The "melting point curve" $\theta_0 yb$ gives the composition of the solid which deposits from any particular liquid when it begins to freeze. Thus a cooling fluid containing x per cent. of B begins to freeze at θ and y is the percentage of B which the solid first deposited contains. At any given temperature (θ), liquid and solid alloys can exist in equilibrium only when their percentage compositions have definite values (represented by x and y respectively). The thermodynamical method of accounting for this condition of equilibrium is referred to later (§§ 12 and 14).

Roozeboom was the first to suggest that the thermal variation of the constitution of cadmium amalgams may be determined by curves like those of fig. 1, so long as the percentage of cadmium does not pass a certain limit (not exceeded in the experiments discussed below), and this suggestion was found to accord with various experimental data obtained by Bijl.

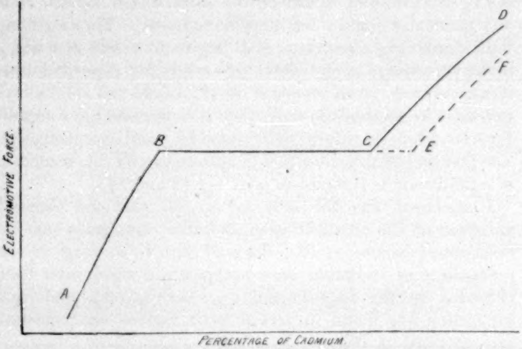
§ 3. *Recent experiments on cadmium amalgams.*—Mr. F. E. Smith has thrown fresh light upon the problem and provided new material for investigation by examining the effect of "chilling," *i. e.* of cooling the amalgams suddenly from temperatures at which they are wholly fluid to a temperature below the freezing point of mercury.

The most striking result of his experiments is shown in fig. 2, which exhibits (diagrammatically) how, at a constant temperature, the E.M.F. of a Weston cell alters with the percentage of Cd in the amalgam. In one set of experiments, represented by the curve ABCD, the amalgam was cooled suddenly (as above) to −50° C. and then allowed to rise in temperature to 0° C. before the cell of which it formed part was constructed. In the other set, represented by ABEF, the amalgam was cooled "slowly" (*i. e.* from the liquid state to 0° C. in several hours) before being used.

On the scale of representation, the curves are identical from A to C, and BCE is a horizontal straight line. The observed time effects and variations in the horizontal parts of the curves are discussed later.

It is easy to anticipate from fig. 1 that the curve for measurements like those of fig. 2 will consist of two branches joined by an intermediate horizontal portion. For at a given temperature θ, any amalgam containing less than x per cent. Cd would be all-liquid, and any amalgam containing more than y per cent. Cd might be all-solid; while intermediate amalgams could consist of mixtures in different proportions of x per cent. liquid and y per cent. solid respectively.

Fig. 2.



In all-liquid and all-solid amalgams the electromotive force would, it is natural to suppose, vary continuously with the percentage of Cd. In mixtures consisting of the same two constituents (in different proportions) it might similarly be expected that the electromotive effect would remain constant.

In this way the general form, either of ABCD or of ACEF, could be accounted for; but the cause of the difference between the two curves is not immediately obvious.

§ 4. *The freezing of slowly-cooled amalgams.*—To understand the exact significance of fig. 2, it is necessary to consider how a mixture, to which fig. 1 applies, freezes.

According to this figure, an alloy containing a per cent.

Cd should begin to freeze at θ_1 and should apparently become solid at θ_2. But, because of the extreme slowness of diffusion in solids, this will happen only when the rate of cooling is so slow that it cannot be dealt with in practice.

At the temperature θ_1 a fluid amalgam containing a per cent. Cd and a solid amalgam containing b per cent. Cd are in equilibrium. When the temperature is lowered slightly a fluid amalgam containing slightly less than a per cent. Cd will be in equilibrium with a solid amalgam containing slightly less than b per cent. A small quantity of the a per cent. amalgam may therefore solidify.

As the temperature falls the percentages of Cd contained by fluid and solid amalgams in equilibrium become continuously lower. The amount of solid material will therefore increase; but the percentage of Cd in the newest crystals will always be less than in those previously formed.

The fluid existing at any given stage of the cooling will be in equilibrium with the solid with which it is in direct contact, but, since most of the successive growths will take place around earlier crystals, this solid will in general enclose older solid, richer in Cd.

There must thus be a continuous diffusion of Cd in the crystallized part of the material from within towards the surface.

In any practical case, where the rate of cooling is not infinitely slow, the diffusion outwards will be very gradual and will not keep pace with the lowering of temperature.

Thus, although (for true equilibrium) an amalgam containing a per cent. Cd should be all-solid just below θ_2, a considerable quantity of liquid, containing c per cent. Cd, will remain. The solid in contact with this liquid will contain a per cent. Cd, but will envelop a considerable quantity of solid richer in Cd than itself, and there will be, in consequence, neither so little c per cent. liquid nor so much a per cent. solid as true equilibrium would imply.

A superior estimate of the amount of liquid remaining at θ_2 can, however, be found. Thus we may imagine that the cooling from θ_1 to θ_2, of an amalgam containing a per cent. Cd, takes place by steps of $d\theta$, and that no diffusion takes place from the solid formed in one step to that formed in the next. Also that the solid formed in any step is of uniform composition and in complete equilibrium with the liquid which remains. Under such conditions, it can be estimated that (in some of the amalgams) about one third of the material might still be fluid at θ_2 although, according to fig. 1, the whole should be solid *.

§ 5. *The effect of sudden cooling.*—From the above sketch of the process of crystallization we see that the surface of a "slowly" cooled amalgam will generally contain a lower percentage of cadmium than the material as a whole, and may even be fluid, although the temperature and percentage composition of the material are such that it should (in true equilibrium) be a uniform solid.

We see also that the relation between the curves ABCD and ABEF of fig. 2 is immediately explicable if, for any reason, the amalgams of the branch CD are of more uniform composition than those of EF.

The former amalgams were cooled suddenly from the fluid state to a temperature much below that at which they would have become completely solid if the rate of cooling had been infinitely slow. Each alloy would therefore pass rapidly through the range of temperature in which equilibrium between two phases is possible, and although, in each element of the material, there might be incipient crystallization with

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* Thus at the end of the first step the temperature is $\theta_1 - \delta\theta$, the liquid phase contains $(a-\delta a)$ per cent. Cd and the solid phase $(b-\delta b)$ per cent. And, of m grams of a per cent. alloy, the quantity

$$\delta m_1 = \frac{\delta a}{(b-a)-\delta(b-a)}\, m$$

will have frozen. It happens that for a considerable range of temperatures and concentrations in the present case, the liquidus and solidus curves of fig. 1 are sufficiently nearly parallel straight lines to permit the assumption, $\delta(b-a)=0$, between θ_1 and θ_2. From this also, if we assume that there are n equal steps of $\delta\theta$ in the cooling process, we get $n\delta a = b-a$, and hence $\delta m_1 = m/n$. The quantity of liquid remaining at the end of the first step is thus $m\left(1-\frac{1}{n}\right)$. Continuing the process it will be found that the quantity of liquid remaining at the end of the nth step is $m\left(1-\frac{1}{n}\right)^n$ which, if we assume n to be very large, has the value $m/2{\cdot}72$ very nearly.

accompanying redistribution of the Cd, as the temperature fell, this process being slow could not proceed very far.

The greater part of the solidification would thus take place at a temperature at which Cd and Hg can exist together in equilibrium only as a homogeneous mixture *. It would therefore occur without redistribution of the Cd with respect to the Hg. What "differential" crystallization there was would be on a scale relatively so minute that the process of equalization by diffusion and the approach to true equilibrium would take place comparatively rapidly as the temperature rose.

There is, therefore, no difficulty in finding a satisfactory general interpretation of fig. 2, as a consequence of fig. 1.

We may now proceed to examine the data more minutely.

§ 6. *Quantitative comparison of the "chilled" and "slowly cooled" amalgams.*—It is reasonable to suppose that two amalgams have the same surface composition when they give the same steady E.M.F. at the same temperature, even although their average compositions may be different.

Hence, if we assume as a first approximation that the values of x and y at any temperature θ (when fig. 1 is taken to represent cadmium amalgams) can be deduced from the thermo-electromotive properties of the series of chilled amalgams of Table VIII. (*l. c.* p. 31), we can determine the state of the surface of any slowly cooled amalgam when the E.M.F. which it gives is known.

For this purpose, some of the data for the slowly-cooled amalgams (Tables I. and II., *l. c.* pp. 18, 19) were plotted, as in fig. 3, along with the data for the chilled amalgams. The dotted curves refer to the former.

Table A, below (p. 377), summarizes the inferences which can be drawn from the figure when the data are interpreted in the way above described.

Each row of numbers gives, for a particular temperature, the surface constituent or constituents of the slowly cooled amalgams (11 to 20 per cent. Cd) deduced by interpolation from the curve, at that temperature, for the chilled amalgams.

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* See § 14 below.

Fig. 3.

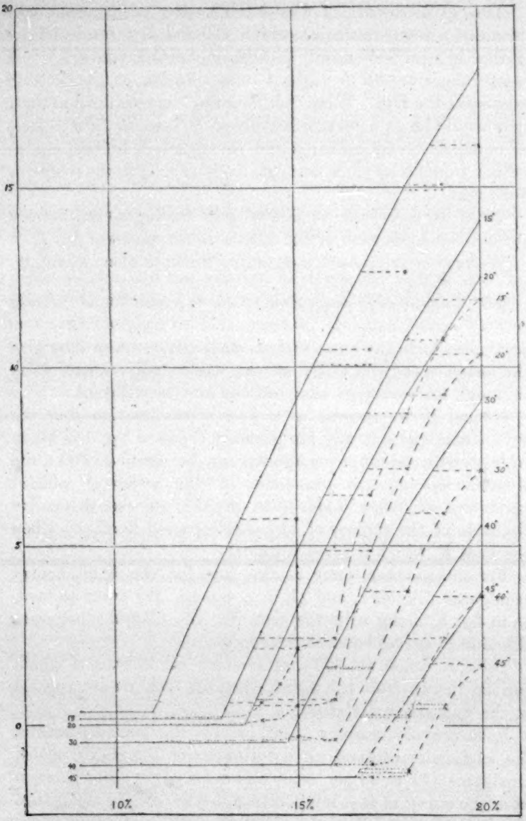

TABLE A.

Mean Percentage of Cd in Amalgam. ➛		11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Temperature. ↓	Time. ↓
Surface Percentages of Cd in Amalgams at ➛ different Temperatures.	i.	12·85 4·3	12·88 (approx.)	12·9 (approx.)	13·1 (approx.)	13·78	13·92	15·75	16·88	17·95	18·65	15°	after about 3 months at 15° C.
	ii.	13·5 4·9	13·52 (approx.)	13·52 (approx.)	13·6 (approx.)	13·77	13·86	15·75	16·9	17·98	18·68	20°	
	iii.	14·7 6·4	14·72 (approx.)	14·73 (approx.)	14·74 (approx.)	14·78 (approx.)	14·75 (approx.)	$15{\cdot}^{73}_{7}$	$16{\cdot}^{83}_{79}$	17·95	18·68	30°	
deduced from Tables I., II., and VIII.	iv.	15·9 7·9	15·9 7·9	15·9 7·9	15·9 7·9	15·9 7·9 (approx.)	15·9 7·9 (approx.)	15·9 7·9 (approx.)	$16{\cdot}^{7}_{68}$	17·95	18·68	40°	
	v.	16·5? 8·8	16·5? 8·8	16·5? 8·8	16·5? 8·8	16·5? 8·8 (approx.)	16·5? 8·8 (approx.)	16·5? 8·8 (approx.)	$16{\cdot}^{67?}_{62?}$	$17{\cdot}^{9}_{83}$	18·65	45°	
deduced from Tables III. & VIII.	vi.	11·0 2·7	11·0 2·7	11·02 (approx.)	11·02 (approx.)	12·83	11·55	15·48	16·77	17·95	18·65	0°	about 2 days after cooling from 45° C.
		11·0 2·7	11·01 (approx.)	11·02 (approx.)	11·22 (approx.)	13·45?	13·2	15·7	16·8	18·01	18·68	0°	about 6 weeks later.

The table begins with the results at 15° C. after the amalgams had stood for three months at this temperature.

(i.) At 15° C. (according to Table VIII.) any amalgam containing less than about 4·3 per cent. Cd should be all liquid, and any containing more than about 12·85 per cent. Cd should be all solid (if in equilibrium). According to Table A, the surface of the 11 per cent. slowly cooled amalgam is a mixture of these alloys. The surfaces of the 12 per cent. and 13 per cent. amalgams appear to be solid alloys containing slightly above 12·85 per cent. Cd. Each of the succeeding amalgams (with the exception of the 16 per cent. amalgam which for some accidental * reason behaves irregularly) is superficially weaker in Cd, in nearly the same proportion (0·92 to 0·94) in each case, than the amalgam as a whole.

(ii.) At 20° C. (to which the cells were next heated) any amalgam containing less than about 4·9 per cent. Cd should be all-liquid, and any containing more than about 13·5 per cent. Cd should be all-solid. The surface of the 11 per cent. amalgam is a mixture of these alloys, and as before the surfaces of the 12 per cent. and 13 per cent. amalgams appear to be solid and to contain slightly more than the percentage of Cd (13·5 per cent.) in the richer constituent of the two-phase amalgam. The surface of the 14 per cent. amalgam contains about 13·6 per cent. Cd.

It will be noticed that the surface concentrations of the richer alloys (15 to 20 per cent.) as deduced by this method are, as nearly as can be measured, the same at 20° as at 15°.

(iii.) At 30° C., the equilibrium amalgams should (according to Table VIII.) contain approximately 6·4 per cent. and 14·7 per cent. Cd. With certain limitations, discussed later in § 9, (which apply equally to similar cases at the other temperatures), the surface of the 11 per cent. amalgam is now a mixture of these alloys. And we might expect that

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* It is obvious that irregularities of cooling and distribution must sometimes occur. That sudden changes of the slopes of the curves, between 15 and 20 per cent. Cd, are due to accidental irregularities is shown by comparison of the present data with those for another series (Table VI. *l. c.*), in which similar changes of slope occur at other percentages.

the surfaces of the 12 to 16 per cent. amalgams would be similarly constituted. But, as in (i.) and (ii.), the surface film in these amalgams seems to be solid, since it contains a slightly higher percentage of Cd than the solid component of the two-phase system.

From the results in column (ii.) we might expect the surfaces of the 17 to 20 per cent. alloys to remain unchanged. It will be seen that this is true of the 19 and 20 per cent. amalgams; but the percentages of Cd in the surfaces of the 17 and 18 per cent. amalgams are beginning to diminish slightly and are exhibiting a time effect.

The significance of these unlooked-for differences is discussed in § 10 below.

(iv.) At 40° C. the limiting alloys contain about 7·9 per cent. and 15·9 per cent. Cd. As we might expect, the surfaces of the 19 and 20 per cent. amalgams remain unchanged. On the other hand, the percentage of Cd in the surface of the 18 per cent. amalgam is continuing to fall. The surface of the 17 per cent. amalgam has become practically a two-phase system.

(v.) At 45° C., the limiting percentages are about 8·8 and 16 5. The surface of the 20 per cent. amalgam remains practically unchanged; but, unexpectedly as before, the surface percentage in the 19 per cent. amalgam is now slowly falling. The decrease at the surface of the 18 per cent. amalgam continues.

(vi.) The amalgams were now cooled to 0° C. After six weeks at this temperature (last row of Table A) the surfaces of all the amalgams containing more than 14 per cent. Cd (excluding the irregular 16 per cent. amalgam) had returned practically to the state in which they were when the measurements at 15°, as in (i.), were made.

Table IX. (*l. c.* p. 32), described as typical, can be analysed (by comparison with Table VIII.) in exactly the same way as Tables I. and II. In connexion with this Table the author writes:—"At temperatures near to but below the first transition temperature the diffusive processes in an unstable amalgam are no doubt accelerated, and the outer shell becomes richer in cadmium with a corresponding increase in the E.M.F. of the cell. . . . ." No doubt rise of temperature will accelerate the diffusion; but it is to be remarked that, as shown in Table A, an "unstable" amalgam which has stood for some time at the ordinary temperature can be raised through 20° or 30° C. without measurable increase in the surface percentage of cadmium, and that the first noticeable effect of temperature rise is a surface decrease of an unexpected kind. The same effect is shown in Table IX.

Sometimes (*e. g.*, *l. c.* pp. 23, 24, and Conclusion 1, p. 38) Mr. F. E. Smith writes as if an unstable amalgam may consist of one central solid mass surrounded by a shell of lower concentration. It seems to me, however, that this state of affairs must be very exceptional.

§ 7. *The probable structure of frozen amalgams.*—It is a well-established characteristic of crystallization in general, that it proceeds around nuclei distributed more or less uniformly throughout the cooling material. The uniformity of distribution of the nuclear growths will be affected in the present case by the fact that the solid grains are of less density than the fluid out of which they separate. The first grains to form must tend to rise. On account of their smallness they will not rise rapidly. As crystallization proceeds a solid net-work will develop within the material. The later crystallization will take place from fluid entangled within the meshes of this net-work. Consequently the material need not in general separate into upper, all-solid, and lower, all-fluid, parts.

In an amalgam which is still partially fluid, but would be all-solid in true equilibrium, at the temperature of observation, the mean concentration of a layer near the surface will no doubt be greater than that of a layer near the bottom. But the difference need not be very marked, and there may be an appreciable quantity of fluid in the spaces between individual grains near the surface, which will take a long time to disappear. If the density effects are appreciable, a greater proportion of the earliest formed crystals will be present in the upper layers. The centres of the grains nearest the surface may then be richer in Cd than the centres of those lower down; but even when the amalgam has stood long enough for the upper portion to become solid throughout, there will be graduations in the percentage of Cd, from point to point in that portion, of which the existence will become obvious (as described below) when the temperature is raised.

§ 8. *An effect of the presence of the electrolyte at the surface of the amalgam.*—The surface layer of a partially crystallized amalgam will consist of a number of grains between which are spaces filled with liquid amalgam. Some of the solid grains will in general project slightly above the mean surface level. The layer of fluid amalgam covering these will be, at most, very thin.

Under the ordinary process of diffusion from the grains the liquid surrounding them will gradually diminish in amount, more or less uniformly in all directions. But it is important to notice that the process of equalization of distribution of Cd will be accelerated, in the surface, when (as in the case of the Weston cell) the amalgam is covered by a solution of cadmium sulphate.

The very thin layers of x per cent. fluid in immediate contact with the surface grains * will soon receive enough Cd by diffusion to convert them into solid containing more than y per cent. Cd (fig. 1). In consequence they will no longer be in electromotive equilibrium with the neighbouring, relatively thick, layers of surface fluid which still contain only x per cent. Cd. Electrolytic action will ensue. Cadmium will enter solution round the surface grains and will be deposited upon the adjacent fluid.

The electrolyte will thus act as a distributor, over the whole surface, of the cadmium diffusing from the surface grains. The surface will thus tend to acquire a thin and probably solid skin which, on account of its lesser density, will have no tendency to sink below the underlying fluid.

In the presence of $CdSO_4$ solution, the whole of the surface film may therefore become solid and of uniform composition, although, on account of the extreme slowness of ordinary intermolecular diffusion, there may still be an appreciable quantity of fluid alloy underneath.

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* When a saturated solution of $CdSO_4$ is poured over the fresh surface of a partially fluid amalgam, the positions of these surface grains can easily be seen.

§ 9. "*Minor irregularities*" *of Weston cells.*—Regarding the surface behaviour of partially fluid amalgams in the above way, it is possible to account for peculiarities, at first sight perplexing, of the horizontal branches of the curves for the chilled and the slowly cooled amalgams. The vertical scale of fig. 3 is not sufficiently open to show these peculiarities clearly. They are exhibited in fig. 4 which represents the behaviour of the amalgams, at various temperatures, over the range represented by the branch BC of fig. 2.

The data for the chilled amalgams are marked by dots and for the slowly cooled amalgams by crosses.

In the slowly cooled amalgams, as in the chilled, the rise of E.M.F. near B is at most very gradual. But, towards C, the former amalgams behave differently. The rise is much greater than in the chilled amalgams, and the E.M.F. of a slowly cooled amalgam is now considerably greater than that of the corresponding chilled amalgam.

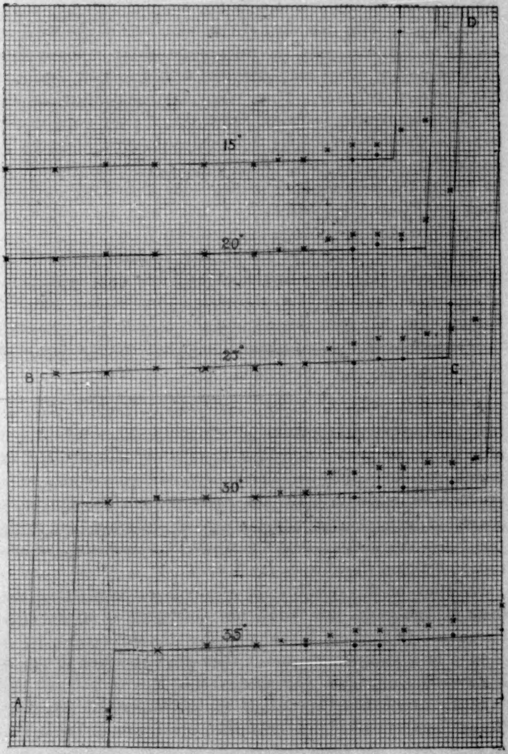
This effect is exhibited numerically in Table A, and has already been alluded to in § 6. It may be explained as follows :—

We assume, for reasons given in § 5, that the solid grains in partially fluid chilled amalgams are of more uniform composition than those in similar slowly cooled amalgams. A chilled amalgam which is nearly all solid at any temperature θ will consist mainly of solid y per cent. Cd amalgam, together with a small amount of fluid containing x per cent. Cd.

A slowly cooled amalgam, of the same average composition, will contain a smaller proportion of solid grains ; but these will be richer in Cd. Their surfaces will contain y per cent. Cd ; but their interiors will contain more. If a sufficient number of these grains be present in the surface originally, or rise thereto after detachment by heat, they may produce a thin surface skin containing more than y per cent., as described in § 8 above. The slowly cooled amalgam will then give a greater E.M.F. than the corresponding chilled amalgam.

The excess over *y* per cent. Cd in the surface will only be appreciable when the surface skin is relatively thick, *i. e.*

Fig. 4.



when the surface grains are relatively numerous. For much of the under surface of the skin will be in contact with liquid x per cent. alloy, and cannot therefore contain more than y per cent. Cd.

Thus it is only in the stronger slowly cooled amalgams that the E.M.F. can be much greater than that of the two-phase alloy—in agreement with fig. 4.

In keeping with this view of the behaviour of the stronger slowly cooled amalgams, the percentage of Cd in the surface skin of any of them may increase within certain limits as the temperature rises. Thus when the temperature of one of these amalgams is raised from θ to θ', the solid grains must partially liquefy. By this means the percentage of Cd in the fluid alloy is raised from x to x', and the percentage of Cd in the new surfaces of the solid grains is y', greater than y. Electrolytic effects may ensue as before and cause the surface skin percentage to rise above y'.

As the temperature is raised the surface grains may become relatively more numerous at first, since additional grains may float up to the surface; but in the end the surface skin will grow thinner and the E.M.F. will approach nearer to that of the two-phase amalgam corresponding with the temperature of observation.

§ 10. *Skin effects in all-solid amalgams.*—It will be obvious that electrolytic skin effects of the kind described in § 8 may occur in amalgams which are already all-solid, but in which ordinary diffusion effects are still proceeding with appreciable velocity. In such cases the surface film may acquire a larger share of the effects of diffusion than areas just below the surface, with the result that these may be less rich in Cd than the surface layer itself.

Evidence of this condition of affairs is revealed when the data of Tables II. and VIII. are compared as in Table A above (see § 6), where it is shown that the surface begins to change at a temperature lower than that at which it would if no weaker amalgam were present near it.

Similar considerations might explain otherwise puzzling time effects exhibited by the chilled amalgams of Table VI. (*l. c.* p. 28). In these, practically without exception, the marked increase of E.M.F. which occurs in the first few days is succeeded by a small but unmistakeable decline. An effect so general cannot be due to accidental irregularities in the process of crystallization.

The statement that "it may be due to the chilling producing a too highly concentrated amalgam in the outer shell" scarcely removes the need for further explanation. Effects of this kind are not confined to the chilled amalgams. The 15, 17, and 19 per cent. amalgams of Table II. show appreciable decline in the surface percentage of Cd during the three months at 15° C.

Another possible cause* of a slow decrease of E.M.F., which should be mentioned, depends upon the deduction (from § 12 below) that the equilibrium concentration of Hg salt in solution round an amalgam must be less than that round pure mercury. In cells of the Weston type, Hg salt must thus be diffusing continuously from cathode to anode and the amount of Hg in the latter must be slowly increasing by precipitation. Fortunately, the diffusion will generally be so slow that its effect upon the E.M.F. of a two-phase amalgam will remain unnoticeable for a very long time. But if the anode surface consists of a thin single-phase skin the percentage of Hg within it may increase perceptibly in a comparatively short period.

§ 11. *The question of the horizontality of BC in fig.* 2.—The skin effects described in § 9 arise primarily from the lightness of the solid grains and from their want of uniformity of composition. As soon as such effects become appreciable, departure from horizontality must ensue. A measure of the importance of these effects is given by the difference between the E.M.F.s of the richer amalgams, chilled and slowly cooled, of fig. 4.

But another question remains. If the amalgams (chilled or otherwise) were of quite uniform composition between B and C, differing only in the relative amounts of x per cent. and of y per cent. amalgams present, would BC be absolutely horizontal, *i. e.*, would the E.M.F.s of all the amalgams within this region be found to be identical, however refined the means of comparison might be?

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* Possible effects at the cathode are outside the range of the present paper.

The data of fig. 4 seem to establish the fact that there is always a slight rise from B towards C. In the region near B the amount of solid amalgam is so small that there is little room for appreciable variation in the percentage of Cd between the centres and surfaces of individual grains. Hence, as the electromotive data show, chilled and slowly cooled amalgams containing the same percentage of Cd must here be of practically identical composition. But even in this region the E.M.F. curve slopes upwards.

It is thus impossible to say beforehand that the E.M.F. does not rise as the percentage of Cd in the amalgam increases, even when the crystals are as uniform as they can possibly be.

§ 12. *The possibility of equilibrium between two amalgams and the same electrolyte.*—The question raised in the preceding section cannot be answered satisfactorily without more careful consideration of the conditions of equilibrium between each amalgam and the electrolyte than has been so far necessary.

It will perhaps be useful to indicate first how, neglecting surface energy, the conditions of equilibrium between the two homogeneous amalgams can be represented thermodynamically *.

The total energy ϵ of a homogeneous substance containing masses m_1 and m_2 of its two components can change by acquisition of heat (alteration of the entropy η), performance of external work (alteration of the volume v) and change of composition (alteration of the mass of either constituent).

For a reversible change we may write

$$d\epsilon = \left(\frac{\partial\epsilon}{\partial\eta}\right)_{vm_1m_2} d\eta + \left(\frac{\partial\epsilon}{\partial v}\right)_{\eta m_1m_2} dv + \left(\frac{\partial\epsilon}{\partial m_1}\right)_{\eta vm_2} dm_1 + \left(\frac{\partial\epsilon}{\partial m_2}\right)_{\eta vm_1} dm_2.$$

The values of the first and second partial differential coefficients are obviously θ and $-p$. The terms containing them represent the energy variation due to change in the heat content and volume of the working substance; the remaining terms indicate how the energy variation depends upon the composition. The partial differential coefficients which they contain are functions of the composition of the working substance and we may write

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* *Cf.* Gibbs, Trans. Conn. Acad. vol. iii. pt. 1, p. 115.

$$d\epsilon = \theta\, d\eta - p\, dv + \mu_1\, dm_1 + \mu_2\, dm_2.$$

For any other homogeneous mixture of the same substances, also capable of reversible variation, we may write

$$d\epsilon' = \theta\, d\eta' - p\, dv' + \mu_1'\, dm_1' + \mu_2'\, dm_2'.$$

Now suppose that these two mixtures can coexist in equilibrium.

By hypothesis the components are independent variables. We may imagine that a small quantity *dm* of the m_1 component leaves the second phase and enters the first in such a way that the transference takes place without change in the entropy or volume of either phase. The energy variation of the system would then be

$$\left(\frac{\partial\epsilon}{\partial m_1}\right)_{\eta v m_2} dm_1 + \left(\frac{\partial\epsilon'}{\partial m_1'}\right)_{\eta' v' m'_2} dm_1' = (\mu_1 - \mu_1')dm.$$

But since neither the entropies nor the volumes change, the system neither does external work nor acquires heat. Consequently we must have

$$\mu_1 - \mu_1' = 0,$$

and, by a similar argument,

$$\mu_2 - \mu_2' = 0.$$

These and two other equations, derived one from each phase at given θ and p, suffice to define the conditions of equilibrium completely—the essential variables being

m_1/v, m_2/v and m_1'/v', m_2'/v' respectively *.

In assuming, as above, that the total energy of a known mass of each phase, at given θ and p, is dependent only on its composition, we neglect the possible influence of surface energy.

We may continue to suppose that, to a first degree of approximation, the surface tension and electrostatic potential difference between the x per cent. and y per cent. amalgams are negligible and proceed to examine the conditions under which the two amalgams could coexist in presence of the same electrolyte.

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* *Cf.* § 14, below.

Considering first the equilibrium of fluid amalgam and electrolyte, reversible exchange of Cd or Hg between electrode and electrolyte will be possible since the latter is a solution of sulphates of Hg and Cd. But while the masses dm_1 and dm_2 of the metals may be considered neutral in the amalgam, they are associated with definite positive charges, which we may write $k_1\, dm_1$ and $k_2\, dm_2$, when in solution.

In order that (as in the equations already given) we may still regard m_1 and m_2 as independent variables, we assume that when a quantity dm_1 of mercury leaves the solution, a quantity dm_3 of anions (SO_4) carrying the charge

$$k_3\, dm_3 = -k_1\, dm_1$$

also leaves the solution and accumulates at the surface separating electrode and electrolyte. We assume also that a similar effect accompanies the transference of dm_2 of cadmium.

In consequence, if we suppose the electric potential of the amalgam to be V and of the electrolyte to be V″, it will be seen that the reversible energy variation may now be written

$$\begin{aligned} d\epsilon + d\epsilon'' = \theta(d\eta + d\eta'') - p(dv + dv'') + (\mu_1 - \mu_1'')dm_1 \\ + (\mu_2 - \mu_2'')dm_2 + (\mathrm{V} - \mathrm{V}'')(k_1\, dm_1 + k_2\, dm_2) \\ + (\bar{\mu}_3 - \mu_3'')\left(\frac{k_1}{k_3}dm_1 + \frac{k_2}{k_3}dm_2\right), \end{aligned}$$

in which $$\left(\frac{\partial\epsilon}{\partial m_1}\right)_{\eta v m_2} = \mu_1 + k_1\mathrm{V} + \frac{k_1}{k_3}\bar{\mu}_3,$$

with similar expressions for the other quantities, $\bar{\mu}_3$ referring to the surface layer and μ_3'' to the interior of the electrolyte.

For equilibrium, as before,

$$\left.\begin{aligned} (\mu_1 - \mu_1'') + k_1(\mathrm{V} - \mathrm{V}'') + \frac{k_1}{k_3}(\bar{\mu}_3 - \mu_3'') = 0, \\ \text{and} \qquad (\mu_2 - \mu_2'') + k_2(\mathrm{V} - \mathrm{V}'') + \frac{k_2}{k_3}(\bar{\mu}_3 - \mu_3'') = 0. \end{aligned}\right\}$$

In the same way, considering the solid amalgam and the electrolyte, we must have

$$\left.\begin{aligned}&(\mu_1'-\mu_1'')+k_1(V'-V'')+\frac{k_1}{k_3}(\bar{\mu}_3'-\mu_3'')=0\\ \text{and}\quad&\\ &(\mu_2'-\mu_2'')+k_2(V'-V'')+\frac{k_2}{k_3}(\bar{\mu}_3'-\mu_3'')=0.\end{aligned}\right\}$$

Considering the equilibrium of the amalgams with each other we assume

$$\mu_1=\mu_1', \quad \mu_2=\mu_2'. \quad \text{and} \quad V=V'.$$

Suppose now that we ignore possible differences at the surfaces separating the electrolyte and the fluid and solid amalgams respectively and put $\mu_3=\bar{\mu}_3'$ (see also § 13, below).

Then it will be possible to find values of μ_1'', μ_2'' and μ_3'', which will be related in such a way as to satisfy the above equations simultaneously and make

$$V-V''=V'-V''.$$

In other words it will be possible to find electrolytes within which the two amalgams can exist side by side in complete equilibrium. With any one of these electrolytes the E.M.F. would be independent of the relative amounts of the two phases present.

§ 13. *The effect of surface energy.*—If, as above, the surface energy variations of the liquid and solid amalgams are regarded as identical, the conditions of equilibrium assume a simplicity which there is no doubt they do not possess.

We do not know the relative importance of the energy per unit area of the surface separating the amalgams; but the difference between the energies of the surfaces separating the respective amalgams and the electrolyte is perceptible. The liquid amalgam appears always to spread over the surface of the solid amalgam in the presence of the electrolyte. We may therefore assume that the surface energy between solid amalgam and electrolyte is greater than that between liquid amalgam and electrolyte.

There are phenomena which show that the relation between surface energy and potential difference is often very complex and § 12 is put forward merely as a method of representing the general nature of what occurs. It can be seen, however, that even if the conditions of equilibrium were as simple as those already given, the difference of surface energy would cause $\bar{\mu}_3'$ to exceed $\bar{\mu}_3$. The two amalgams could not then, subject to the condition $V-V''=V'-V''$, be in equilibrium with the same electrolyte.

The value of $V''-V'$ satisfying the second pair of equations would be greater than that of $V''-V$ satisfying the first two, *i. e.*, the potential of a given electrolyte with respect to the solid amalgam would exceed that of the same electrolyte with respect to the fluid amalgam. Hence, supposing $V=V'$, the amalgams could not coexist in contact with the same electrolyte.

Such considerations are enough to show that there is no theoretical necessity for horizontality of BC. It is much more likely that the equilibrium at the anode is between a variable phase and the electrolyte, and that the E.M.F. electrolyte/electrode is distinctly greater when the amalgam is one which is just on the point of partial liquefaction at θ than when it is one in which the last traces of solid have just disappeared.

There is therefore every probability * that the E.M.F. of a cell of the Weston type must always be to some extent dependent upon the percentage of Cd in the amalgam.

§ 14. *The application of the phase rule to cadmium amalgams.*—The considerations given at the beginning of § 12 were introduced in order to explain a method of regarding the effects of surface energy. They also supply a means of interpreting the "equilibrium curves" of fig. 1. As in § 12, if two homogeneous phases containing components A and B can coexist in equilibrium we must have not only equality of temperature and pressure, but also two other equalities which we may write $\mu_a=\mu_a'$ and $\mu_b=\mu_b'$.

To determine completely, at given θ and p, the state of any mixture of A and B, we require to know how these

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* Mr. F. E. Smith has kindly supplied me with further details of some of the measurements recorded in his Table VIII. which confirm this view.

"potentials" vary with c, the percentage of B in the mixture.

It is conceivable that the substances can form a series of mixtures in all proportions and that any one of these can be entirely fluid or entirely solid at the temperature θ. For a homogeneous mixture, at constant temperature and pressure, we must have

$$m_a d\mu_a + m_b d\mu_b = 0.$$

Considering all-fluid mixtures first we may suppose, following Gibbs *, that, when c approximates to zero, μ_b must have a very large negative value, whilst μ_a is finite and $d\mu_a/dc$ has a finite negative value. Similarly, when c approximates to 100, μ_a must have a very large negative value, while μ_b is finite and $d\mu_b/dc$ has a finite positive value.

The curves connecting the variations of μ_a and μ_b with c may no doubt be complicated; but, in the simplest cases, they may (from what precedes) take forms like AM and BN of fig. 5 (p. 392).

We may assume that analogous relations, represented by the curves A′M′ and B′N′, hold for the all-solid mixtures.

If, as in fig. 5, a temperature is chosen which is below the freezing point of B, but above the freezing point of A, we shall have $\mu_b > \mu_b'$ for the phases of pure B, and hence B will lie above B′; but $\mu_a' > \mu_a$ for the phases containing pure A, so that A′ will be above A.

The conditions for coexistence of a fluid phase containing x per cent. B and a solid phase containing y per cent. B are

$$[\mu_a]_{c=x} = [\mu_a']_{c=y}$$

and

$$[\mu_b]_{c=x} = [\mu_b']_{c=y}.$$

Inspection of fig. 5 will suffice to show the possibility of finding values of x and y which satisfy these conditions.

The same conclusion can be reached, less directly, by considering the variations with c of the total thermodynamical potentials ζ and ζ' per 100 grams of all-fluid and all-solid mixtures, remembering the relations

$$d\zeta/dc = \mu_b - \mu_a \quad \text{and} \quad d\zeta'/dc = \mu_b' - \mu_a'.$$

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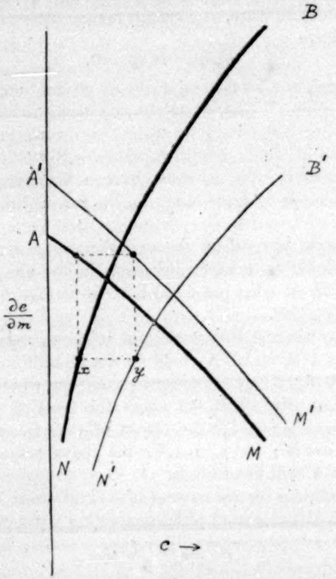
* *L. c.* p. 194

In the system of fig. 5, therefore, coexistence of two phases is possible when the liquid phase contains x per cent. B and the solid phase y per cent. B. All mixtures containing between

Fig. 5.



x per cent. and y per cent. of B, and only those, can split into two phases.

If, however, the temperature is below the freezing point of A, as well as below that of B, the relative positions of AM and A′M′ will be reversed and it will now be impossible to find values of x and y for which μ_a, μ_a' and μ_b, μ_b' respectively, are equal. In other words, one-phase solid mixtures only will be stable *.

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* *Cf.* § 5 above.

If it is assumed that the μc variations for the mixtures of Hg and Cd used in Weston cells are of the form represented (diagrammatically) in fig. 5, it can at once be seen why the amalgams exhibit the properties which have been discussed.

§ 15. *The temperature coefficients of cadmium-mercury cells.*—The efficiency of the Weston cell as a standard does not depend only on the fact that the chemical composition of the amalgam can vary within considerable limits without producing more than a few millionths of a volt difference in the electromotive force ; but also upon the extreme smallness of the effect of temperature change near 0° C.

It is instructive to consider how the existence of the two-phase amalgams happens to be the cause of the second phenomenon.

The data of Table XI. (*l. c.* p. 36) show that the smallness of the temperature coefficient is not due to absence of heat exchanges with the surroundings during isothermal working of a two-phase cell, but to the fact that these nearly balance each other. Somewhere near 5° C. there is an exact balance and the temperature coefficient vanishes.

Above 5° C., the temperature coefficient has a small negative value. This means that slightly more heat escapes from the cell (during action) than it absorbs from the outside. The presence of the two-phase amalgam reduces the net loss of heat. For, during the working of the cell, the average percentage of Cd in the anode must diminish and the equilibrium compositions of the coexisting phases can only be maintained by reduction in the amount of the richer (solid) phase. This will occasion absorption of heat and so reduce the net amount evolved.

Below 5° C., the absorption just mentioned slightly overbalances the remaining effects and the temperature coefficient is positive.

Abstract.

In this paper the recent experiments of Mr. F. E. Smith on cadmium amalgams are discussed from the point of view of the modern theory of alloys. Theory and experiment alike suggest that there is no range at any temperature over which the E.M.F. of a Weston cell is absolutely

independent of the percentage of Cd in the amalgam. Even if the materials are quite pure, the existence of surface energy must cause some variation. Within the range over which the E.M.F. is usually taken as constant the E.M.F. appears to rise, very slowly, with increase in the cadmium content. The rate varies, but is never more than a few millionths of a volt for one per cent. Cd. From the data it seems possible also to discover the precise way in which the use of the richer two phase amalgams may lead to variability of the E.M.F. of the Weston cell. The interpretation advocated is that the irregularities are due to electrolytic skin effects arising out of want of uniformity of composition of the surface grains. The probable reason why the temperature coefficient of E.M.F. of a Weston cell, always small, actually vanishes near 0° C. is indicated. An outline of the way in which the phase rule can be used to exhibit the nature of the incomplete equilibrium of the Weston cell is also given.

DISCUSSION.

Mr. RAYNER referred to hysteresis effects which, he said, were a source of serious trouble when working with Weston cells. He gave details of the behaviour of a particular cell during a period in which the temperature of the cell was raised from 14° C. to 25° C. for three days, and asked the Author if he could explain the observed variations of the E.M.F. on the theory he had advanced.

The AUTHOR, in reply to Mr. Rayner, said that, while he was not prepared (without further information) to give a fixed opinion concerning the behaviour of particular cells, he thought he had shown how to find the exact causes of any peculiarities for which pure amalgams might be responsible. The utility of the theory he had described lay in the fact that it should be possible, by means of it, to discover what the effect on the amalgam of any particular thermal treatment would be. His object had been to discuss one cause of some of the variations of the Weston cell, and to indicate a theoretical limit to the accuracy obtainable.

XXVII. *A Galvanometer for Alternate Current Circuits.*

By W. E. SUMPNER, *D.Sc., and* W. C. S. PHILLIPS, *B.Sc.*[*]

THE steady electromotive forces and highly sensitive galvanometers available for use with direct current tests render such tests excellent whether deflexional or balance methods are in question. Tests involving change of current, such as induction measurements, are not so satisfactory. In ballistic tests the best galvanometers are in many cases not sensitive enough, and though balance methods are available they are usually not so simple in working as those in which steady currents are employed. Alternate current tests are still less satisfactory. It is impossible to generate an electromotive force whose constancy is comparable with that of a battery or accumulator. Special difficulties arise owing to effects of frequency, wave-form and phase. Balance methods can be devised, but they are rarely of much use, owing to the above difficulties and to the absence of sensitive instruments. A distinct advance has been made during the last few years by the construction of improved forms of vibration galvanometer. But this instrument overcomes only some of the difficulties. It is a sensitive indicator rather than a measuring instrument. It must be adjusted to resonance for the best effects, and its sensitiveness is necessarily affected by slight changes in current frequency.

The sensitiveness of ballistic galvanometer tests can be greatly increased by the use of mechanical commutators such as the secohmmeter of Ayrton and Perry, or subsequent modifications of this by Fleming and Lyle. But such methods only make use of a crude form of alternate current, and it appears that the simplest and most effective cumulative method for testing effects due to changes of current, must in the end prove to be one involving the use of alternate currents generated in the ordinary way. The construction of the indicating instrument constitutes the real obstacle. The reflecting instrument for alternate current circuits described in this paper is the result of an attempt to overcome this difficulty. The instrument is like a moving coil galvanometer in almost every respect, except that its field is due to an electromagnet excited by an alternating voltage. The theory of non-reflecting instruments of the same type has already been fully explained *, but it may be convenient to briefly refer to it.

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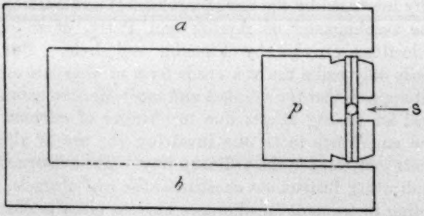
[*] Read June 10, 1910.

If an alternating voltage V be applied to an electromagnet whose winding consists of m turns, the core flux N will be such that

$$V = rA + m\dot{N}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where r is the resistance of the coil and A the current traversing it. If the coil and electromagnet be so designed that for currents of the frequencies used the resistance is negligible in comparison with the impedance, we can neglect the term rA. It follows that the rate of change of N will be at each instant a measure of V, and this will be true whatever the permeability or hysteresis of the core. We thus have a magnet whose strength is accurately determined by the applied voltage whatever the physical properties of the core. Such an electromagnet can easily be made very strong. Moreover, the shorter the air-gap between the poles is made, the denser is the magnetic flux due to a given current, and the greater is the ratio of impedance to resistance for a given winding. Thus the stronger the electromagnet is made by improving its magnetic circuit, the more accurate it becomes, provided this electromagnet is excited by the voltage of the circuit.

Fig. 1.



The instrument here described has a laminated electromagnet formed of stampings shaped like figure 1. These

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* Proc. Roy. Soc. vol. lxxx. (1908), "Alternate Current Measurement."

stampings are of two kinds—a rectangular portion with two straight limbs, *a*, *b*, forming the core of the electromagnet, and a specially shaped stamping, *p*, between the poles. The stamping, *p*, is separated from the magnet limbs on each side by an air-gap. Each gap consists of two portions, a narrow part about 1 mm. across bounded by the straight edges of *p*, and a wider portion in which one of the vertical sides of the moving coil can turn round the curved edges of *p*. The moving coil, of 50 turns, is similar in shape, suspension, and mode of control to that of an ordinary permanent magnet instrument. It has a central spindle indicated at *s* and working in a recess suitably stamped in *p*. The pile of stampings is about 4 cm. deep and 9 cm. long, the limbs being 1 cm. wide. Each of the limbs, *a*, *b*, is wound with a coil of 2000 turns of fine wire, and also with a coil of 100 turns of thicker wire. The two fine wire coils are put in series and connected to three terminals; the two thicker coils are also put in series and joined up to two additional terminals. Thus the instrument can be excited by a winding consisting of either 200, 2000, or 4000 turns, according to the voltage used. The iron is not too strongly magnetized if the winding used contains 20 turns per volt on 50 cycle circuits. Thus 200 volts may be applied to the 4000 turn coil, or 10 volts to the 200 turn coil. But the instrument is so sensitive that such excitation will only be needed for exceptional tests. The moving coil may be used with a condenser or other apparatus, either on some special circuit or in conjunction with one or other of the field coil windings. The instrument has been constructed by Robt. W. Paul, to whom several of the working details are due. The following are some of the uses :—

Use as a Voltmeter.

If a voltage V be applied to a field coil of *m* turns and if another voltage V_1 be applied through a condenser of K microfarads to the moving coil, it can be shown that the torque acting on the moving coil is a measure of

$$\frac{1}{m} \mathrm{K} \, \overline{\mathrm{V}\mathrm{V}_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or of the mean product of the two voltages. If the condenser voltage is obtained from one of the field coils of n turns the torque is measured by

$$Kn\left(\frac{V}{m}\right)^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

There is really another factor the value of which would not be quite constant if the induction density in the gap varied with the position of the moving coil. But in the present instrument this factor is essentially constant owing partly to the shape of gap adopted, and partly to the fact that for a reflecting instrument the movement of the coil is very slight. Numerous tests have shown that for any given choice of coils and condenser, the scale deflexion is strictly proportional to the square of the applied voltage quite up to the limits of the scale used (300 mm. each side of zero for a scale distance of 1 metre).

The numbers denoted by m and n may each be chosen either 200, 2000, or 4000, while the capacity K may be given widely different values. It is thus clear that the instrument can be used as a voltmeter for a large number of ranges. It will be sufficient to indicate two of these. From a number of tests made under various conditions, the value of expression (3) when V is measured in volts and K in microfarads is found to be $1{\cdot}6 \times 10^{-4}$ for a scale deflexion of 200 millimetres. It follows that this deflexion can be obtained

for 200 volts if $m = 4000$, $n = 200$, and $K = 3{\cdot}2 \times 10^{-4}$ m.f.,

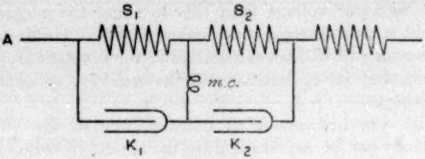
or for 20 millivolts if $m = 200$, $n = 4000$, and $K = 4{\cdot}0$ m.f.

The deflexion is independent of frequency and wave-form if the field winding to which the voltage is applied has a resistance negligible in comparison with its impedance. This will always be the case if the frequencies used are high. But if the frequency is low and the mass of copper used in the magnetizing coil is small, the resistance of this coil will become comparable with its impedance. The deflexion will then be dependent on frequency, though for a given frequency it will still be a measure of the product KV^2. Thus if a coil of two turns be wound round the core of the magnet and be used as the exciting winding ($m=2$), and if the moving coil be used with the same condenser and field winding as in the second case above ($n=4000$, $K=4$), a deflexion of 200 mm. will correspond with a reactive voltage in the two-turn coil of only 0·2 of a millivolt. But it will be necessary to apply a much greater voltage than this to cause the magnetizing current to flow through the resistance of the winding. The instrument will still act as a voltmeter for constant frequency circuits, but its indications will be sensitive to change of frequency.

With the instrument as actually wound, the effect of frequency can be represented by the measured values of the quantity (3) for a deflexion of 200 mm. If this quantity when multiplied by 10,000 be called Q, then for the arrangement $m=4000$, $n=200$, the value of Q is 1·62 for 50 cycle circuits, 1·61 for 100 cycle circuits, and 1·73 for 25 cycle circuits. That is, the deflexion for a given value of V^2 is essentially the same for all frequencies above 50 cycles per second, but is 6½ per cent. less if the frequency is dropped to 25 cycles per second. For the arrangement $m=200$, $n=4000$, a change of frequency produces greater effect. The deflexion for a given value of V^2 is 5·2 per cent. greater for 100 cycles, and 18·6 per cent. less for 25 cycles, than it is for 50 cycles. When the main coil is used for both voltage and condenser ($m=n=4000$) there is no appreciable change of constant for frequencies between these limits.

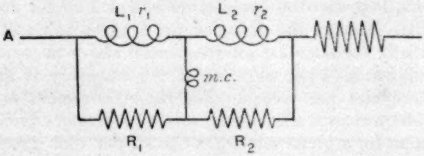
The effect of frequency on the value of Q is mainly due to the phase error represented approximately by the ratio of resistance to impedance of the coil to which the voltage is applied. This ratio on 50 cycle circuits is 2·6 per cent. for the 4000 turn coil, and 10·6 per cent. for the 200 turn coil. But Q is also affected by slight amounts of magnetic leakage (between the windings) dependent on the arrangement of coils used. Other properties of the magnet are deducible from the data that on 50 cycle circuits the power factor of the magnetizing coil is 0·14; the ampere turns needed for the magnet are $6u$; and the flux density in the iron is $80u$; where u is the number representing the voltage applied per 1000 turns, or the millivolts per turn. For special uses of the instrument the phase error of the magnet can be reduced by applying a suitable condenser direct to one of the field windings. Thus the power factor of the 200 turn coil can be raised to unity on 50 cycle circuits by applying a condenser

Fig. 2.



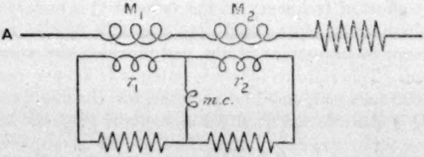
$$K_1S_1 = K_2S_2.$$

Fig. 3.



$$\frac{L_1}{R_1+r_1} = \frac{L_2}{R_2+r_2}.$$

Fig. 4.



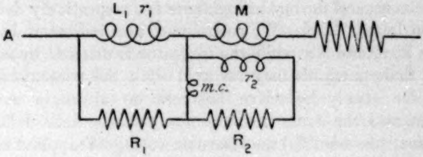
$$\frac{M_1}{R_1+r_1} = \frac{M_2}{R_2+r_2}.$$

of 1·1 m.f. to the 4000 turn coil, the ratio of resistance to impedance is reduced from 10·6 per cent. to 1·5 per cent., and the phase error is reduced to zero.

Use with Null Methods.

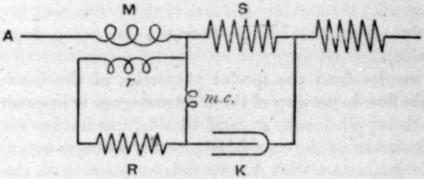
Figs. 2–7 illustrate the ordinary bridge methods for comparing inductances and capacities. In these methods a ballistic galvanometer is used as an indicator, and to test the balance the current A through the arms of the bridge is made or broken by a key. The equation representing the condition

Fig. 5.



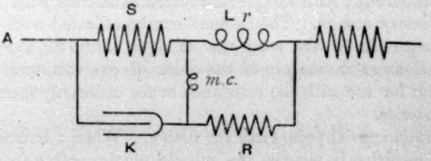
$$\frac{L}{R_1+r_1}=\frac{L}{R_2+r_2}$$

Fig. 6.



$$\frac{M}{R+r}=KS.$$

Fig. 7.



$$KS=\frac{L}{R+r}.$$

for inductive balance is indicated in each case beside the figure. The zero deflexion condition for steady currents holds necessarily in the three cases of figs. 2, 4 and 6; but a troublesome special adjustment is needed in the case of fig 3; while in the cases of figs. 5 and 7 the adjustment for steady currents is impossible. In these figures capacities, self-inductances, and mutual inductances are respectively denoted by the letters K, L, M; resistances are indicated by the letters R, *r*, and S; while the indicator is denoted by *m.c.*

All these methods may be used with the present instrument for steady inductive balances on alternate current circuits, and the same formulæ apply to the zero deflexion condition, provided (i.) the alternate voltage V applied to the field-coil of the instrument also causes the current A through the bridge conductors, (ii.) the alternate current A is made essentially cophasal with V by the use of suitable non-inductive resistances as indicated by the zigzag lines in the figures, (iii.) the moving coil *m.c.* of the instrument is placed directly across the bridge (using a reversing key when desirable).

It results from the special properties of the instrument that the flux in the gap of the electromagnet is in quadrature with the applied voltage (and thus in quadrature with A). The inductances or capacities produce voltages or currents also in quadrature with A, and thus in phase with the flux, so that their phase is such that they produce the maximum torque on the moving system.

These methods have all been thoroughly tested on alternate current circuits with the present instrument, and with most satisfactory results. The balance can be adjusted with ease to one part in 10,000, when the voltages set up on the coils or condensers are merely of the order of one volt, and thus suitable for use with the resistance boxes ordinarily found in laboratories.

Certain special points call for notice. When a balance of great precision is needed, the minute electromotive force *e*, induced in the moving coil by the alternating field of the magnet, tends to cause a small deflexion disturbing the balance. When the moving coil circuit is essentially non-inductive, as for the cases of figs. 3, 4 and 5, the current due to e will be in phase with e, and in quadrature with the flux, and in such cases the corresponding deflexion will in general be negligible. For the inductive circuits represented in figs. 2, 6 and 7, this will not be the case, and a small deflexion due to e will occur. But in all cases any effect due to e can be accurately eliminated by using a false zero method, that is, by adjusting the balance till the reading on the scale is unaltered by switching the bridge current A on or off. The induced voltage e is due to the voltage applied to the field coil, and is unaffected by changes in A. In most cases it will be found sufficiently accurate to take the mean of the two conditions of balance obtained by using a reversing key with the moving coil. The false zero method is simpler and is mathematically accurate, though in practice, as with all false zero methods, there is a liability to a small error due to the variations of the false zero deflexion in sympathy with fluctuations in the main current or voltage.

The formula given for balance expresses the necessary and sufficient condition that the two electromotive forces set up in the coils, or on the condensers, of the bridge, send, through the moving coil, currents which are equal in magnitude and exactly opposite in phase*. But if this condition is not quite fulfilled, the unbalanced current will not necessarily be in the best phase to influence the deflexion unless certain limitations are borne in mind. The resistances R must not be made too small, and the resistances S must not be made too large. Otherwise the sensitiveness of the instrument to indicate want of balance is adversely affected, although the condition of balance remains as stated, except for minute correction terms, due to secondary effects of self-induction, &c., which have been neglected.

* This is strictly true for the cases of figs. 2, 4, and 6. In the cases of figs. 3, 5, and 7 an additional current through the moving coil is caused by the resistance of the inductance coils. This current is in quadrature with the field, and does not cause any deflexion. There is thus no need for a troublesome double adjustment as in corresponding tests in which other instruments are used.

We have found on investigation that the only cases which need be considered are those in which condensers are used (figs. 2, 6 and 7). The value of KSp (where p is 2π times the frequency) represents the tangent of the angle by which the phase of the moving coil current differs from that of the magnet field. It may easily become comparable with unity, as will be apparent from the fact that on 50 cycle circuits with K equal to 1 microfarad, and S equal to 1000 ohms, the value of KSp is 0·314. But it will be found easy to adjust the conditions of the bridge in all the cases considered so as to render these tests quite satisfactory in practice.

One or two examples of these bridge methods may be given to illustrate the conditions of working.

The method of fig. 6 was used to test the values of M for a primary coil of 500 turns in conjunction with two secondary coils. The three coils were wound on a wooden bobbin and the primary wire was suitable for a current of 2 amperes. A current of 1·1 ampere was passed through the bridge, and 28 volts were applied to the 4000 turn coil, the frequency being 50 cycles per second. A standard resistance of 0·9995 ohms was used for S, and a standard mica condenser of 1·0155 m.f. was used for K. An ordinary resistance-box was used for R. Using the first secondary coil the value of $R + r$ was adjusted to 4225·1 ohms. The corresponding value of M works out to be 4·2885 millihenries. A similar test with the other secondary yielded 3·8499 m.h., and one with the two secondaries in series yielded 8·1398 m.h. The sum of the values of M for the two secondaries is 8·1384 m.h. The small inconsistency is easily attributable to errors in the resistance-box, to small capacity effects in these resistances, or to similar causes of no present interest, the point being that it was possible to adjust R to one part in 40,000, under conditions of test which could easily have been rendered more sensitive. The current used with the bridge could have been quadrupled, and the strength of the field could have been increased ten times, without injuring the apparatus, and without altering the quantities under test.

The method of fig. 2 was used to compare the capacity of a paraffin paper condenser (about 0·9 m.f.) with the standard mica condenser just referred to. The 4000 turn field-coil was subjected to 20 volts on a 50 cycle circuit. From this voltage was obtained, by means of a small transformer, a cophase voltage of 2·5 volts suitable for the bridge conductors. The resistances S were kept below 2000, and various tests were made. It was always possible to adjust the balance to 1 part in 10,000, but the inconsistencies in the various tests amounted to 2 parts in 1000 and are attributable to phenomena (such as a partial conduction in the paper condenser) affecting the exact formula for balance.

An adjustable air-condenser, formed of a fixed and moving set of plates like a multicellular voltmeter, was tested for capacity against the standard mica condenser above referred to. The maximum capacity was measured as 0·0023 m.f., or only 0·2 per cent. of that of the standard. It was always possible to measure the capacity in any position far more accurately than the condenser could be adjusted to this position. The method of fig. 2 was used, but higher voltages were applied to the bridge than in the previous test. The 4000 turn field-coil was subjected to 30 volts, and this voltage was also used for the condenser bridge, a resistance of 9000 ohms, made for the pressure circuit of a wattmeter and suitable for high voltages, being used in association with the air-condenser. The standard mica condenser was used as K_1. The associated resistance S_1 was taken from an ordinary resistance-box. It was set at various values up to 22 ohms as a maximum. The 9000 ohm resistance was used for S_2, and the air-condenser K_2 was adjusted for each value of S_1 till balance was obtained. The maximum voltage to which S_1 was subjected in these tests was less than 0·1 volt. The false zero method was employed for balance, the moving coil voltage causing a deflexion of about 10 centimetres.

It is to be noted that in all these bridge tests the phase error of the electromagnet due to the resistance of the field winding does not lead to any error, but merely causes a negligible change of sensitiveness due to a shift of phase of the moving coil current as compared with that of the field.

Other Uses of the Instrument.

The voltmeter tests previously described show that the instrument can be used to measure very small capacities, especially in cases where it is possible to apply high voltages to the condenser. It will be apparent from expression (2) that if the voltage V_1 applied to the condenser is in phase with the voltage V applied to the field, and a known multiple of it, the sensitiveness can be indefinitely increased by making V_1 large.

Thus two circular brass plates of 7·3 cm. diameter and about 6 mm. apart were tested as a condenser, and found to have a capacity of 6×10^{-6} microfarad. This capacity caused a deflexion of 180 mm. when 102·4 volts were applied to the 4000 turn coil, and, by means of a transformer, 890 volts were applied to the condenser plates through the moving coil. The capacity tested being so small it was found necessary to eliminate capacity effects associated with the wires used for the connexions. This was done by taking the difference of two deflexions obtained with the connexion to one of the plates alternately made and broken. Earthing conditions had to be carefully attended to. Good values have been obtained for the specific inductive capacities of plates of various dielectrics, but we have as yet not had time to properly carry out such tests, which for accurate results require balance methods with guard-ring condensers.

The instrument has not yet been tested with alternate currents of higher frequencies than 100 cycles per second, but there appears no reason to suppose there will be any difficulty in the way of its use for high frequency work.

Added July, 1910.—Mr. A. Campbell has drawn our attention to a paper, previously unknown to us, in which Stroud and Oates (Phil. Mag. 1903) describe an instrument resembling in some respects the galvanometer here referred to. The paper gives data of the electromagmet showing that it contained a greater volume of iron than that of the instrument here described. But no details are given of the air-gap between the poles, and it does not appear that any attempt was made to produce a magnetic field whose phase is essentially in quadrature with that of the applied voltage. The characteristics of the present instrument are the result of such a relationship. In the Stroud galvanometer the field in the gap is probably stronger and less uniform than that of the instrument here described ; since the disturbing influence of this field on the moving system seems much more serious.

ABSTRACT.

Tests of effects due to change of current, such as induction phenomena, are often hard to carry out because the ballistic galvanometer available is not sensitive enough. Alternate current tests are still less satisfactory owing to special difficulties. The vibration galvanometer overcomes only a few of these difficulties. It must be adjusted to resonance for the best effects, and its indications vary with current frequency. The sensitiveness of ballistic galvanometer tests can be greatly increased by the use of some form of mechanical commutator by means of which a crude form of alternate current is produced. A better method would be to generate the current in the usual way if a suitable instrument existed. The galvanometer here described is the result of an attempt to construct a measuring instrument by means of which inductances and capacities can be compared by bridge methods as accurately as it is possible to compare resistances. The instrument is like a moving coil galvanometer in almost every respect, except that its field is due to a specially constructed electromagnet excited by an alternating voltage. This voltage V is applied to a winding of m turns of the electromagnet, and the core flux N is such that

$$V = rA + m\dot{N},$$

where r is the resistance of the winding and A the current traversing it. The coil and electromagnet are so designed that for currents of the frequencies used the value of rA is negligible in comparison with V. The rate of change of N will therefore be at each instant a measure of V, whatever the permeability or hysteresis of the core. The instrument has a laminated electromagnet formed of stampings of two kinds—a rectangular portion with two straight limbs forming the core of the electromagnet, and a specially shaped stamping between the poles. The moving coil of 50 turns swings in a narrow gap separating the stampings, in much the same way as in a permanent magnet instrument. On the limbs of the magnet are windings of 200, 2000, and 4000 turns. The iron will not be too strongly magnetised if the winding used contains 20 turns per volt on 50 ∼ circuits, but the instrument is so sensitive that such excitation will only be needed for exceptional tests.

If a voltage V be applied to one of the field coils of m turns, and if the same, or another, field winding of n turns be joined up, through a

condenser of K microfarads, to the moving coil, the torque acting on this moving coil will be a measure of

$$Kn\left(\frac{V}{m}\right)^2;$$

i.e. the deflexion is proportional to the square of the voltage. By suitably choosing K, m and n, the voltmeter may be used over a large number of ranges. Thus with the instrument shown a deflexion of 200 mm. on a scale at 1 metre distance, can be obtained either for 200 volts or for 20 millivolts. The deflexion is independent of frequenev and wave-form if the field-winding to which the voltage is applied has a resistance negligible in comparison with its impedance. Thus with $m=4000$ and $n=200$ it was found that the value of V^2 required to give a certain deflexion was independent of frequency between 50∼ and 100∼, but at 25∼ it was 6½ per cent. less than at 50∼. At any fixed frequency the deflexion will always measure KV^2. The instrument may be used with great advantage to compare inductances and capacities by the ordinary bridge methods, the working conditions being (i.) the alternating voltage V applied to the field coil of the instrument must also cause the current in the bridge conductors, (ii.) the alternate current in the bridge must be made in phase with the voltage V by the use of suitable non-inductive resistances, (iii.) the moving coil must be placed directly across the bridge.

The balance can be adjusted with ease to 1 part in 10,000 when the voltages set up on the coils or condensers are of the order of 1 volt. When a balance of great precision is needed, the minute electromotive force e induced in the moving coil by the alternating field of the magnet, tends to cause a small deflexion disturbing the balance. When the moving coil circuit is non-inductive, the current due to e will be in phase with e and in quadrature with the flux so that the corresponding deflexion will be negligible. But in all cases any effect due to e can be accurately eliminated by working to a false zero. As illustrations of the behaviour of the instrument, the results of tests are given on the measurement of the mutual induction of coils, the comparison of capacities, and the measurement of Specific Inductive capacity.

DISCUSSION.

Mr. A. CAMPBELL expressed his admiration of the galvanometer and in particular of the ingenious method of connecting it with a condenser when voltage is to be measured. He remarked that the instrument was very similar in construction and use to that used by earlier experimenters. Stroud and Oates (Phil. Mag. 1903) described such a galvanometer and showed how sensitive it was for testing condensers by Anderson's method and for other purposes; and Terry (Phys. Review, 1905) used a similar one for accurate comparisons of condensers. Abraham (*Comptes Rendus*, Apr. 1906) described another in which the false zero was got rid of by a compensating arrangement.

With regard to vibration galvanometers, there is no difficulty in keeping them in tune on any reasonably steady alternating circuit. In the bridge methods illustrated in the paper, a double adjustment is necessary when a vibration galvanometer is used, and in general an evaluation of effective resistance or leakage resistance can be made simultaneously with the measurement of inductance or capacity. He asked the Authors if this double adjustment can be entirely dispensed with when the quadrature galvanometer is used. It would be interesting also to know how the instrument would work with non-sinusoidal waveforms in a case where the balance depended on the frequency.

Mr. W. DUDDELL asked if the Authors could supply more numerical data so that the instrument could be compared with other galvanometers of similar design. He pointed out that the tests were simplified owing to a double adjustment being unnecessary.

Mr. IRWIN thought that with small modifications the instrument would be useful for tests other than those described in the paper. It might for example be used in determining the resistance of electrolytes. He pointed out that errors might arise due to a small amount of magnetic material in the moving coil.

Dr. SUMPNER stated, in reply to Mr. Campbell and Mr. Duddell, that the object of the special shape of pole-piece was to increase the impedance of the exciting coil. Electromagnet galvanometers were not new, but the special mode of controlling the magnetic flux by the applied voltage was novel. The behaviour of the instrument was a direct consequence of this device. The vibration galvanometer responded to a voltage irrespective of its phase, and hence in some tests troublesome double adjustments were needed. This was not the case with the new instrument in regard to any of the tests described in the paper. The current taken by the instrument could easily be calculated from data given in the paper. In reply to Mr. Irwin, the Author said that the small deflexion due to magnetic impurities in the moving coil could be eliminated in all balance tests by using a false zero method. In deflexional methods the effect could be rendered negligible by reducing the voltage exciting the field, and using a moving coil current of correspondingly increased strength.

Mr. PHILLIPS stated that the present paper was preliminary to a more detailed account of the work which had been done with the instrument. In reply to Mr. Irwin, he stated that it was proposed to make use of the instrument to measure the Specific Inductive capacity of materials and the resistance of electrolytes.

XXVIII. *Positive Electrification due to Heating Aluminium Phosphate.* *By* A. E. Garrett, *B.Sc.*[*]

[Plate VI.]

I. *Introduction and Experimental Arrangements.*

In a paper published in the 'Philosophical Magazine' for October, 1904, by Dr. R. S. Willows and myself, it was found that the halogen compounds of zinc when heated are able to discharge both positively and negatively electrified bodies. A more detailed examination of this phenomenon was subsequently carried out by one of us [†].

In those experiments the temperature to which the salts were raised was in no case higher than 360° C., and no series of observations at pressures lower than a few mm. were undertaken.

Sir J. J. Thomson [‡] made some experiments to determine whether the base or the acid is instrumental in producing the ionization, and came to the conclusion that the nature of the ionic charge is determined by the acid. Thus he found that phosphates when heated produce a very large excess of positive ions, halogen compounds produce an excess of positive ions, and nitrates an excess of positive at first, but when heated sufficiently to be converted into oxides they produce an excess of negative. Incidentally he found that aluminium phosphate gives off a very large excess of positive ions.

Now the halogen and other compounds used in the previous experiments are known to be bodies which undergo decomposition when strongly heated, and the ionization in this case may be due to chemical action. Aluminium phosphate, on the other hand, is an extremely stable substance, and it is of interest to investigate the source of ionization in this instance. This salt was therefore chosen for detailed examination.

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[*] Read June 10, 1910.

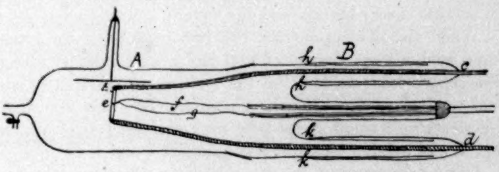
[†] Garrett, Phil. Mag., June 1907.

[‡] Cambridge Phil. Soc. Proc., p. 105, 1907.

The apparatus used is shown in the following diagram.

The glass tube consists of two parts connected by a ground-glass joint. The part A is shown in vertical section, the part B in horizontal section. *e* is the strip of thin platinum foil, $\frac{1}{4}$ sq. cm. in area, on which the salt is placed to be heated; the temperature of the platinum *e* is raised by a current, the leads for which are the thick copper wires *c*, *d*. The electrode E, which is connected through a galvanometer to earth, is an oblong aluminium disk about 3 sq. cm. in area; the distance between the heated salt and this electrode was in most cases 0·5 cm. *f*, *g* are wires of the thermo-electric couple used for ascertaining the temperature of the platinum foil with which they are fused. To obtain the temperature from the thermo-electric current, the deflexion of the galvanometer to which leads from the junction pass was noted when a tiny particle of K_2SO_4 just melted on the foil. In this way the deflexion for two temperatures differing by about 1000° C. was obtained, and the temperature in degrees centigrade corresponding to any other deflexion could be got by aid of the correction curves given by Callendar *. The reliability of this method was checked by observing the melting-point of Na_2SO_4. The observations of the behaviour of aluminium phosphate were taken over a range of temperature from 900° C. to 1300° C.

Fig. 1.



Before commencing work with aluminium phosphate it was ascertained that only a small current due to ions of either sign could be detected when the platinum foil was used alone.

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* Phil. Mag. [5] xlviii. pp. 519 *et seq.*

The sensitivity of the galvanometer was such that a deflexion of one scale division represented a current of 2×10^{-9} ampere.

When taking readings, a saturation voltage obtained from a battery of small accumulators the negative terminal of which was earthed, was put on *e* (fig. 1) as soon as the heating of the salt was commenced, and the deflexion of the galvanometer noted from time to time. When the voltage was taken off *e* in order to read the thermo-current, time was allowed, when the voltage was again put on, for the current to become steady before readings were taken.

When the heating-current had been continued for some time, such a large amount of heat had been conducted along the copper leads as to cause the melting of the wax which was used to render the tube air-tight. This was remedied by soldering a small metal tube over each lead as shown in fig. 1, and making air-tight wax joints at *h* and *k*.

During the course of the experiments in which the galvanometer was used, many attempts under varying conditions were made to detect the presence of negative ions, but with no success. That such ions are present was afterwards proved, but from the results obtained they must be less than $\frac{1}{2}$ per cent. of the positive.

Now if the positive ions are produced by chemical change brought about by the heat, then one would expect that a decay in the current would take place with the time. It was found that the current does decrease when heating is continued, so it was decided to obtain the curve of decay for this substance.

II. *Decay of Positive Ionization.*

The aluminium phosphate was made into a paste with distilled water and then placed on the foil *e*; the foil was heated slightly by the current before putting it in the tube, this causes the phosphate to adhere to the platinum, and also gets rid of the excess of water. To lessen the effect of the contained gas the tube was evacuated, and the temperature of the foil quickly raised to the degree desired. Varying conditions of temperature and pressure were tested, and it was found that the most expedient method was to reduce the pressure to about 0·05 cm., and to use a temperature of about 1200° C. Under such circumstances it was possible to obtain a record of the decay in 5 or 6 hours.

The method of procedure was as follows :— + 60 volts were put on *e* (fig. 1), and as soon as the temperature became steady, readings were taken ; the pressure and temperature were kept constant throughout.

Experiments of this kind were carried out with air, hydrogen, and CO_2, as the gas in the tube. It was found in every case when air or hydrogen was used, that the manner in which the current varied with the time for the first 20 minutes was peculiar.

A typical curve to illustrate this is given (Pl. VI. fig. 2). In this curve the currents are plotted as ordinates and the times as abscissæ.

This shows a rapid fall from A to B, then a rise to a maximum at C, and lastly a decay C to D, in which the current decreases roughly exponentially with the time. The part of the curve near B sometimes showed still further irregularities.

With CO_2 as the gas in the tube, a typical decay curve is represented by ECD. The initial changes observed in air, and hydrogen, are apparently due to water, since a preliminary heating of the phosphate at a lower temperature sufficient to expel the water, gets rid of them altogether.

Further, in the case of CO_2, that gas may possibly assist in the removal of hygroscopic moisture, and so prevent its action on the salt.

After some hours' heating, a more or less steady state was reached. This state persisted for some months, nor did it regain any activity if dry or moist air was admitted, even if left for 2 or 3 days. Only on one occasion was a slight temporary regain noted, and in this case the interval was 16 days.

If after the steady state was reached the salt was moistened with distilled water, there was a large increase in the current which quickly died away, and in about 10 minutes the steady state was again reached. The decay of this, AF (Pl. VI. fig. 2), being taken in conjunction with a typical decay curve ECD, a curve of the form ABCD is obtained.

This is further evidence that the initial changes are due to water. These changes will not be further considered.

A typical decay-curve over a longer period omitting these changes is shown in Pl. VI. fig. 3.

The part of the curve near C is exponential; the portion EDC is very similar to the curve given by Rutherford in the 2nd edition of 'Radioactivity,' p. 342, for the variation in activity of the active deposit of Th due to a very short exposure to the emanation. In the case of Rutherford's curve, this is known to be due to the decay of two substances, Th A and Th B, the former of which does not produce any rays, and the equation of the curve is of the form $A(e^{-\lambda_1 t}-e^{-\lambda_2 t})$. The curve of decay of aluminium phosphate can be represented by an equation of the same form up to the point C, beyond this the exponential curve lies below the experimental. Neglecting this latter point for the moment, if the physical analogy as well as the algebraical one holds, we can suppose the first effect of the high temperature is to produce from the phosphate and the surrounding gas a substance A which does not emit ions, that this next produces a substance B which is the agent producing the conductivity. As in Rutherford's case, the curve alone does not allow one to say to which substance the different λ's refer.

If the analogy holds further and the salt also independently produces C, the latter supplying ions but decaying so slowly that its decay can be neglected, the activity due to C is represented by a curve having an equation of the form $B(1-e^{-\lambda_3 t})$, and the whole curve EDCAB should be capable of being fitted by an equation

$$A(e^{-\lambda_1 t}-e^{-\lambda_2 t})+B(1-e^{-\lambda_3 t}).$$

This is actually found to be the case. The values of the λ's depend of course upon the temperature of the salt.

In Pl. VI. fig. 3 the dots represent experimental points, the $\odot$'s points calculated from the above formula.

It can be seen from the curve, that immediately after the very rapid decay there is a somewhat steady state which is in turn succeeded by a gradual rise in the current to a final steady state. Owing to the small scale used, this is not well shown in the curve at AB, although in the experiment there illustrated the actual rise amounted to about 20 per cent. of the previous steady values. The upper curve AB represents this section on a larger scale and makes this point more distinct.

The relatively large current at the beginning of the heating appears to depend very largely upon the nature of the gas in the tube, while the final steady current is due entirely to the heated salt.

The results obtained with hydrogen were always of a most irregular nature. The rate of decay was also much slower in that gas than in air or CO_2. It was quicker in CO_2 than in air.

No alteration of the gas contained makes any difference to the current when in the steady condition if the pressure and temperature are kept constant. Further, although an increased current is obtained with higher temperatures, and an alteration in the pressure produces also an alteration in the current, yet on bringing both temperature and pressure to their former values, the same steady current is obtained.

III. *Attempts to increase the Activity.*

Richardson * found that a platinum wire which is heated to such a temperature that an excess of positive ions are given off, gradually becomes less and less active in this respect until a more or less steady state is reached. When the wire reaches this state he found that it could be rendered much more active by passing, for about a minute, an electric discharge through the tube containing the wire, and he further showed that this regained activity persisted for a considerable time after the discharge had ceased.

It was thought that a discharge might have a similar effect upon the heated phosphate when it had reached the steady state. The heating-current was therefore cut off and a discharge from a Ruhmkorff coil sent through the tube. This increased the current temporarily. If the coil was again applied less effect was produced, until, after several repetitions, the coil produced no effect.

The actual period during which the discharge lasted was gradually increased, and it was found that the maximum effect was attained when the discharge continued for ½ minute. On no occasion did it require more than 20 minutes again to reach the original steady state after discontinuing the discharge. In order to ascertain whether the temporary increase in current was due to the effect which the discharge produced upon the salt itself, or upon the residual gas, a fresh supply of CO_2 was admitted when the state at which the coil discharge produced no effect was reached, and the tube pumped down to 0·12 cm. pressure. The coil discharge again caused a temporary increase. The direction of the coil discharge did not influence the result.

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* Phil. Mag. [6] pp. 93 *et seq.*, 1903.

The extra current therefore appears to be due to an effect produced by the action of the discharge upon the surrounding gas.

IV. *Action of Heat in the absence of Electrostatic Field.*

When the electrode *e* was insulated while the heating was continued, an abnormally large current was obtained so soon as the field was put on between *e* and E (fig. 1). This current gradually decayed, but it was some 2 or 3 minutes before it reached its normal value. The magnitude of the increased current was found to depend upon the time during which *e* was insulated. It gradually increased with the time, and reached a maximum when the insulation had lasted ten minutes. Any longer period of insulation was found to have no further increasing effect upon the current.

It is possible that during the time of insulation positive ions are being freed in the salt on '*e*,' these accumulate in the substance and are prevented from escaping by a discontinuity of the potential at the surface. This continues until, at the end of 10 minutes, the field due to the accumulated charges is sufficient to take them over as they are formed, when of course no further increase in the number accumulated takes place.

When the field is again put on, these ions are of course dragged out, and the initial value of the increased current will depend upon the number of ions which have accumulated.

This effect greatly increased the labour of taking readings under varying conditions, as considerable time had to be allowed after a change had been made for the current to become steady. These experiments were carried out after the steady state had been reached.

This effect was more apparent in some gases than in others, of those tried it was most marked in the case of CO_2.

V. *Effect of Pressure upon Current.*

When the salt is in the steady state it is most convenient for investigating the changes due to alterations of pressure and temperature.

The changes in the current when the temperature was kept constant while the pressure was varied were first undertaken. The contained gases used were air and CO_2. The results obtained were of a similar nature for both these gases.

Some typical curves are shown in the accompanying diagram (Pl. VI. fig. 4), in which the pressures are plotted as abscissæ, and the currents as ordinates.

From these it can be seen that starting from zero pressure there is a very rapid rise in the current in all cases, the higher the temperature the more rapid being the rise. For each temperature the current reaches a well-marked maximum value. This, again, is more pronounced at the higher temperatures. When the maximum current has been obtained, any further increase in the pressure produces an immediate and rapid decrease in the current. For the temperatures investigated, this decrease continues until the pressure attains the value 5 or 6 cms., after this the decrease which takes place in the current for any further increase in the pressure is always smaller but is still quite marked. The rate of decrease is always more rapid at the higher temperatures.

When the pressures for which the currents have maximum values are plotted against the temperatures, it appears as if the pressure and temperature are connected by a straight-line law, and that if the temperature could be pushed to a sufficiently high degree the largest current would be obtained in the highest attainable vacuum.

The fact that the maximum currents obtained at the higher temperatures and lower pressures were always greater than those obtained with the lower temperatures and higher pressures is further evidence in support of this view.

It is quite possible, however, that the slope of the temperature-pressure line may gradually become less and less, and that it never actually reaches the zero pressure line.

The peculiar manner in which the current varies with the pressure under these conditions cannot be explained by the collision of moving ions, as in all cases the voltage used was that corresponding to the flat part of the saturation curve. Evidence given later suggests that neutral doublets, such as Righi suggested compose the magneto-cathode rays, and Sir J. J. Thomson * found indications of in his experiments on positive electricity, are driven off when the salt is heated, and the current may be due to the break up of these doublets by collision with the gas molecules.

These doublets would be shot off with relatively large velocities at the lower pressures, but would have small chances of colliding with gas molecules or other doublets, and so, relatively few free ions are produced. At higher pressures the doublets would have much slower speeds but greater chances of collision. Somewhere between these extremes the most favourable conditions for obtaining a maximum current may be looked for at each temperature.

At the higher temperatures the velocity of ejection is greater, and a less number of collisions will be required in order to split up the doublet, hence the maximum current is obtained under such conditions at a lower pressure.

With an increase of pressure more collisions are possible, but less doublets escape from the heated salt since the temperature is lower, hence the current obtained is smaller.

The great drawback in the way of this explanation is the very small current due to negative ions, as compared with that carried by positive ions, which can be obtained under these conditions.

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* Phil. Mag. xviii. pp. 828 *et seq.*, Dec. 1909.

VI. *Effect of Temperature at Constant Pressure.—Phosphate in Steady State.*

Richardson * has proved that the formula $I = a\theta^{\frac{1}{2}}e^{-Q/2\theta}$, where I = saturation current, and θ = the temperature in degrees absolute, while Q = a measure of the energy associated with the liberation of an ion, represents the connexion between the saturation current and the temperature, for positive as well as for negative ions given off by heated platinum wires.

The same law holds for various chemical compounds which have been tested up to a temperature of 360° C. about †.

The following results have been obtained by heating aluminium phosphate in CO_2 at 0·05 mm. pressure. Different quantities of the salt having been used in these experiments, the absolute values of the currents are not comparable.

I.

Temperature, ° C.	Current, 2×10^{-9} amp. as Unit.
880	1
950	4
970	7
995	15
1030	35
1055	49
1095	103
1110	126

II.

Temperature, ° C.	Current, 2×10^{-9} amp. as Unit.
1036	1
1088	3
1135	5
1160	8
1195	15
1230	34
1245	35
1295	74
1330	168

The diagram (Pl. VI. fig. 5) shows the results obtained when $1/\theta$ is plotted against $\frac{1}{2}\log_e\theta - \log_e I$. The points are fairly evenly distributed about straight lines, and these lines are parallel to one another. Thus the two sets of readings are in accord with one another, and the law may be looked upon as true for positive ions given off by heated aluminium phosphate in an atmosphere of CO_2, up to a temperature of 1300° C.

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* Roy. Soc. Phil. Trans. A. 207, pp. 22 *et seq.*

† Garrett, Phil. Mag. June 1907 pp. 732 *et seq.*

When the tube was filled with hydrogen gas at a pressure of 0·05 mm., and the results obtained were plotted in a similar manner, the same law was found to hold for that gas up to 1300° C. (see Pl. VI. fig. 5, curve iii.).

The value of Q may be obtained direct from the diagrams, by multiplying the tangent of the angle which the line makes with the $1/\theta$ axis by 2.

By this means it is found that for the temperatures ranging from 900° C. to 1300° C. when the surrounding gas is CO_2 at a pressure of 0·05 mm., the value of Q is $7{\cdot}1 \times 10^4$, while with the hydrogen gas under similar conditions of pressure the value of Q is only $5{\cdot}3 \times 10^4$ for temperatures ranging from 1095° C. to 1300° C.

VII. *Determination of e/m.*

Sir J. J. Thomson's cycloid method* was used for this purpose. In this method the ions move in a gas at very low pressures under the influence of a magnetic and electrostatic field acting at right angles to one another. For a given electrostatic field, the magnetic field was altered until it caused an appreciable diminution of the current passing to the electrode E (fig. 1). These experiments could not be pushed far because a magnetic field of sufficient strength could not be created. The magnet used produced a field of 800 gausses. The distance between the electrodes was 0·45 cm., and the air pressure in the tube was less than 0·01 mm. When the lower electrode was at a positive potential of 6·3 volts, the magnetic field produced a decrease of about 10 per cent. in the current.

Thomson has shown that in the case of ions starting from a given plane, the value of e/m for these ions may be found from the formula $e/m = 2V/H^2d^2$, in which V is the voltage to which the electrode from which the ions start is raised,

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* 'Conduction of Electricity through Gases,' 1st edition, pp. 107 *et seq.*

H the value of the magnetic field, and "*d*" the distance between the electrodes in cms.

Substituting the above experimental values in this equation we find that $e/m = 9700$ about. Similar values were obtained from other experiments.

This value of e/m refers, of course, to the lightest positive ions present.

Thomson found for the positive ions from hot platinum, values ranging from 60 to 720.

The value of e/m for the hydrogen atom in electrolysis is taken as 10^4. From this it is seen that these positive ions are comparable in size with the hydrogen atom, if we assume the same value for "*e*" in both cases.

A quite appreciable though smaller diminution of the current was also obtained under such conditions that e/m when calculated was found to be some 3 or 4 times as large as given above. Either the ions affected in this case have a mass smaller than that of the hydrogen atom, or else their velocity is much less than that due to the electrostatic field applied; in the latter case we might assume that they started as free ions at some point between the two electrodes, and not as such from the surface of the heated salt.

VIII. *Velocity with which ions are shot off from the Salt.*

Earlier in the paper it has been suggested that some of the ions escape from the salt on account of their kinetic energy without the application of an electrostatic field.

To put this in evidence a Dolezalek electrometer was substituted for the galvanometer, since the ions so escaping could not be detected with the latter. The sensitivity of the electrometer was such that 1 volt produces a deflexion of 180 scale-divisions. The pressure was reduced to 0·01 mm. and the temperature varied as required.

A weak magnetic field was imposed so as to remove any effect which negative ions might produce, and both E and the lower heated electrode are earthed initially.

E was then found to receive a positive charge on disconnecting it with earth.

If the lower electrode was now given a negative potential, this had to be raised to 1·2 volts to stop altogether the charging up of E.

If V = the negative potential to which the electrode *e* is raised to prevent positive ions leaving it, *e* = the charge on an ion, *m* = its mass, and *v* = its velocity; then, from $Ve = \frac{1}{2}mv^2$ we can calculate the velocity "*v*" with which these ions are ejected from the heated salt.

Taking the value of e/m obtained earlier, we find that $v = 1 \cdot 4 \times 10^6$ cm. per sec., a velocity comparable with that (10^7) of the positively charged particles which constitute the anode rays.

With weak magnetic field of too low a value to affect the positive ions, the charging up effect was in every case increased, the final deflexion of the electrometer being always greater when the field was on.

When, however, a field of 800, which had previously been found to produce a measurable decrease in the current due to positive ions was used, it was found that with temperatures below 1050° C. about, the rate of charging up of the quadrants was diminished when the field was on. As the temperature was reduced below this limit, the effect of the field became more marked. This was tested to temperatures about 950° C.

Above 1050° C. the magnetic field caused an increase in the rate at which the electrometer was charged up by the positive ions, and when the temperature had reached 1200° C. about, the rate with the field on was twice as rapid as when the field did not act.

These increases in the rate of charging up can be explained by the fact that negative as well as positive ions are produced under these conditions. These ions would cause the rate at which E charged up to be smaller than if positive ions alone were present, so when they are prevented by the magnetic field from reaching the electrode the rate at which it charges up increases. This, however, does not occur until the number of negative ions which are deflected is in excess of the number of deflected positive ions, and this state is apparently not reached until the temperature is above 1050° C.

It must be understood that the actual number of ions present under these conditions was very much less than in those cases in which the galvanometer was used, since no indication of ions of either sign was then obtained in the absence of an electrostatic field.

From 1050° C. to 1200° C. the rate at which E was charged up in the absence of the magnetic field was practically constant, while the rate with the field on gradually increased.

This would occur if the actual excess of positives which reached E per second remained constant, and for this to be the case, since more ions are now present, negative and positive ions must be formed in equal quantities, such as might happen when neutral doublets split up.

It may be that salts which give off an excess of positive ions when heated, at first eject positive ions only. Next, it would appear that doublets are ejected also, and when the salt is raised to a positive potential, or is at zero potential, the quickly moving positive ions may by collision with the doublets, cause them to break up, hence causing a large positive current to pass between the electrodes. The initial positive ions appear to increase in the number given off per second as the temperature is raised until about 1050° C., after which the output appears to remain constant.

If, on the other hand, the salt is raised to a negative potential, the initial positive ions are prevented from leaving it, and so at reduced pressures, with electrodes a very small distance apart, the doublets have smaller chances of breaking up; thus the current when the salt is negatively charged is relatively small compared with that due to positive ions.

That the potential to which the salt is raised has an important bearing upon the relative number of positive and negative ions present is clearly shown by the fact that the current due to negative ions is not so small compared with that due to positive ions when the salt is at zero potential and the gas pressure is low.

Also, at atmospheric pressure when the products due to heating the salt were removed by an air blast, there was always quite a large current produced by the negative ions.

IX. *Nature of the Ions at Atmospheric Pressure.*

For this purpose the apparatus used was two brass tubes of 1·7 cm. internal diameter, insulated from each other, and each having an insulated wire 0·33 cm. diameter and 28 cm. in length along the axis. These wires could be in turn connected with a Dolezalek electrometer of such sensitivity that one volt caused a deflexion of 820 scale-divisions.

The method adopted was to heat the phosphate on platinum foil through which a current was passed, and to suck air past the heated salt, and then through the two brass tubes placed one behind the other.

In order to avoid great fluctuations in the temperature, the platinum and its leads were enclosed in a wide glass tube, one end of which was connected air-tight to the testing apparatus, and the other closed loosely with cotton-wool.

Readings were taken after the salt had been heated for a sufficiently long time to bring it to the steady condition.

The electrometer showed no initial leak even when one of the electrodes was connected with its quadrants, and the tube containing the electrode was raised to a positive potential of 650 volts. A saturation current was obtained when the electrode nearer the heated salt was connected with the electrometer and a potential difference of 314 volts was used. This is shown by the following Table :—

Volts.	Current due to positive ions in Arbitrary Units.	Volts.	Current due to positive ions in Arbitrary Units.
42	46	390	130
84	58	432	130
126	75	474	128
168	85	516	130
210	95	558	126
236	111	608	130
275	123	656	130
314	130		
348	130		

Under the conditions of experiment (velocity of air through the tube being 40 cms. per sec.) it can easily be calculated from the formula

$$v=\frac{(b^2-a^2)\log_e b/a}{2Vt},$$

in which

v=velocity of ions in cms. per sec.,
b=radius of the tube,
a=radius of wire electrode,
V=potential-difference in volts between wire and tube,
t=time taken by air to pass from one end of the electrode to the other,

that all ions with velocity greater than 0·0027 cm. per sec. are withdrawn when the saturation voltage of 314 volts is put on. When the voltage is raised to 656, ions must have a velocity less than 0·001 cm. per sec. to be able to escape from the first tube.

If the front electrode (*i. e.* one nearer the heated salt) is earthed, and the front tube kept at any positive potential greater than 320 volts, no current should be found near the back electrode when this is connected with the electrometer, and the back tube raised to any positive potential ; a current could, however, be detected.

Even when both tubes were raised to a positive potential of 656 volts there was a current of 8 or 9 arbitrary units. The total current on the back electrode when the front tube was earthed was found to be 118 units. Thus about 8 per cent. of the total current here appears to be due to ions which are too slow moving to be extracted even with the very high voltage used, or which have been formed after passing the front electrode.

It may be mentioned that it was necessary to raise the back tube to a high potential before any indication of these extra ions was forthcoming. Thus with 432 volts on the front tube, a current could be just detected when 190 volts was put on the back tube.

If the current of 8 units on the back electrode is due to ions which are too slow to be stopped by the field on the front electrode, then some indication of these would be expected on the saturation current curve, *i. e.* after passing 314 volts the curve should still gradually ascend.

Nothing of the kind, however, takes place, and, as is seen from the table of observations given, a small increase could easily have been detected.

If we assume that neutral doublets are present which break up into ions after passing the first electrode, then the results are readily explained.

Should the extra ions be in reality due to the splitting up of neutral doublets rather than to the presence of ions of extremely low velocity, one would expect to find an equal number of negative and positive ions formed after passing the first electrode, and this should occur although the saturation currents on the front electrode due to positive and negative ions may differ widely. For aluminium phosphate these saturation currents on the front electrode are in fact very different. It was found that a negative voltage of 220 on the front tube was sufficient to obtain a saturation current with negative ions. When both tubes were brought to a negative potential of 656 volts, the front electrode being earthed, and the back electrode connected with the electrometer, there was a current of 8 units—*i. e.*, precisely the same as that obtained with positive ions under exactly similar conditions.

This appears strong evidence in favour of the view that the extra ions are due to the splitting up of neutral doublets and not to ions of very low velocity.

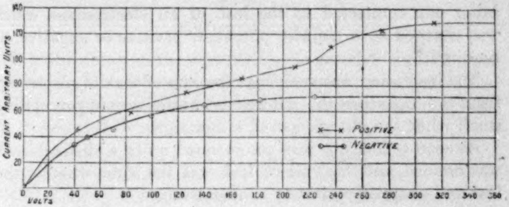
When a current voltage curve for the positive ions is plotted using the values given above, it is seen that the curve formed by joining the points obtained is not of the usual type.

At about 200 volts there are indications that the current is nearly saturated, yet on slightly increasing the voltage, it again rises very rapidly and does not actually reach its saturation value until over 300 volts are put on.

On the other hand, the curve obtained with negative ions is quite a smooth one and does not show any indication of saturation at a voltage lower than 220, nor does any further increase in current take place if the voltage is raised from 220 to 656. This is shown in fig. 6, in which the upper curve represents the current due to positive ions, the lower curve that due to negative ions at different voltages.

This, taken in conjunction with the fact that a current is obtained at the second electrode when 656 volts are put on the first, suggests that either two classes of positive ions are present, or else neutral doublets. The former is rendered unlikely on account of the constancy of the current which passes between the electrodes after 320 volts is reached.

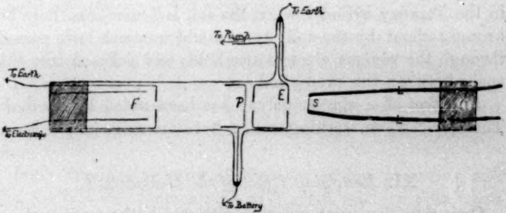
Fig. 6.



X. *Further Evidence of Doublets.*

In order to test whether neutral pairs as well as ions are shot off from the heated salt when surrounded by gas at a low pressure, the apparatus shown in the next diagram was used.

Fig. 7.



The essential part of this apparatus consists of a long glass tube, in one end of which is a Faraday cylinder F, and in the other end the heated salt on the platinum foil S.

During the experiment the foil was kept at a positive potential sufficiently high to prevent negative ions from leaving the salt. The wire gauze P was also kept at such a positive potential as to drive back any positive ions which have passed through the earthed metal tube E.

The apparatus was evacuated to an air-pressure of about 0·01 mm. The outer Faraday cylinder was earthed and the inner one connected to the leaf of an electroscope which was charged to a definite potential, positive or negative, as required.

The insulation was such that the rate of leak of electricity from the leaf when the salt was not heated was imperceptibly small when the charge was of either sign.

As soon as heating was commenced quite a distinct leak was noticed, and the rate of leak was the same whether the leaf was charged positively or negatively.

On the other hand, after the inner cylinder had been reduced to zero potential, no charging up could be detected, thus showing that there was no excess of free ions of either sign in the neighbourhood of the inner cylinder.

No positive ions could pass through the field between P and E, and it is highly improbable that any free negative ions leave the salt when the latter is charged positively, as in the experiment. Again, the field between P and the outer cylinder is a further preventive against any stray negative ions passing through the small hole into the space between the cylinders. Hence it seems that any ionization produced in the Faraday cylinder when the salt is heated, can only be brought about by the split up of doublets which have passed through the various electrostatic fields and diffused into the space between the two cylinders.

An effect of a similar nature has been noted by Sir J. J. Thomson * when working with a hot lime cathode.

XI. *Rectifying Effect of the Heated Salt.*

Owing to the great velocity with which the positive ions are shot off from the heated salt when the pressure is very low, it was thought that such a tube as shown in fig. 1 could be used for rectifying alternating currents providing the temperature of the salt was kept within certain limits. It has been found by various experiments that it can be so used.

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* Phil. Mag. Dec. 1909, pp. 829 *et seq.*

Further work on this point is in progress.

XII. *Summary.*

(i.) The decay of the current due to positive ions obtained by heating aluminium phosphate has been investigated, and it is found that the curve connecting current and time can be represented by a formula of the type

$$A(e^{-\lambda_1 t}-e^{-\lambda_2 t})+B(1-e^{-\lambda_3 t}).$$

(ii.) During the first part of the decay, the nature of the surrounding gas and the water contained by the salt have an important influence. When the steady state is reached the gas has no apparent influence, but water still temporarily increases the activity.

(iii.) The discharge produced by an induction-coil temporarily increases the current which is carried by the positive ions.

(iv.) When the temperature is kept constant, it is found that for each temperature there is a definite pressure at which the current is a maximum. This pressure being lower, the higher the temperature.

(v.) The Richardson formula $I=a\theta^{\frac{1}{2}}e^{-Q/2\theta}$ can be used to express the relationship between the current and absolute temperature when the pressure is kept constant.

(vi.) A value is obtained for *e/m* which indicates that the smallest positive ions present at the lowest pressures must be of a magnitude comparable with that of the hydrogen atom.

(vii.) The high velocity of the ions at low pressures, and also the fact that some escape with great velocity even when no external field is applied, leads one to expect that a tube in which some aluminium phosphate is heated might be of use as a rectifier for alternating currents. It can be so used.

(viii.) The experiments (*a*) with varying pressures at constant temperature, (*b*) at atmospheric pressure in which a current is produced after all the ions are apparently removed, and (*c*) in which the charge on a Faraday cylinder leaks away when care is taken to prevent free ions reaching it, seem to indicate that neutral doublets as well as free ions are ejected from the salt.

In conclusion, I should like to thank Dr. R. S. Willows, in whose laboratory these experiments have been carried out, for the interest he has taken throughout the course of this research, and Mr. F. C. G. Bratt for help in the construction of the apparatus used.

Cass Technical Institute,
Jewry Street, E.C.
May 1910.

ABSTRACT.

In 1904 Dr. R. S. Willows and the Author communicated to this Society the results of some experiments on the halogen compounds of zinc, in which it was shown that those compounds when heated ionized the air around so that both positively and negatively electrified bodies gradually lost their charge. Work in this direction was continued, and the results obtained showed that a large number of inorganic compounds possess properties of a like nature to the above. In 1907 Sir J. J. Thomson found that chlorides, phosphates, and nitrates give off an excess of positive ions when heated, and he incidentally discovered that aluminium phosphate was most active in this direction. As many of the substances previously examined are known to be unstable (and hence the ionization produced may be the result of chemical change), while aluminium phosphate on the other hand is a most stable substance, it was thought that an investigation of the ion producing properties of that compound would be of interest. The apparatus was arranged so that the salt could be heated to the desired temperature (900°–1300° C.) on a strip of platinum foil. The pressure could be reduced as required. For most of the experiments a sensitive galvanometer was used as recording instrument. With the usual distance between the electrodes (0·5 cm.) it was found that a difference of potential of 60 volts was sufficient to obtain a saturation current with the positive ions. The way in which the current varied with the time when the saturation voltage was applied was first investigated. For the first half-hour the current was somewhat irregular. This was found to be due to the water present in the salt. The current for the first hour or two appears to be largely influenced by the surrounding gas. Neglecting the preliminary effects due to water,

the decay of the current with the time can be represented by a curve having the general formula

$$A(e^{-\lambda_1 t}-e^{-\lambda_2 t})+B(1-e^{-\lambda_3 t}).$$

The current finally obtained depends almost entirely upon the salt itself. After 5 or 6 hours heating, no further change takes place in the current obtained under fixed conditions of temperature and pressure. A temporary increase, however, can be brought about by moistening the salt, and also by passing a discharge from an induction-coil through the tube for about ½ min. When the salt is insulated and heated for some time, an extraordinarily large current passes when the field is first put on. This effect increases with the time up to about 10 minutes, after which any longer insulation causes no increase of the initial current. With the salt in the final steady state the conditions were suitable for experiments on temperature and pressure effects. It was found that for every temperature tried there was a certain pressure at which the current obtained was a maximum, and the higher the temperature the lower was this pressure. The currents obtained with constant pressure and variable temperature indicate that the relationship between the rate of production of positive ions and the absolute temperature can be represented by the Richardson formula $a\theta^{\frac{1}{2}}e^{-\frac{Q}{2\theta}}$. By using a quadrant electrometer in place of the galvanometer, it was found at pressures of 0·01 mm. that some positive ions are ejected with a velocity of the order 10^6 cm. per sec.; and other experiments showed that the ions at very low pressures and acted upon by small electrostatic fields moved with great velocity, so it was thought that this substance could be made use of as a means for rectifying alternate currents. It was found that it could be so used. The values of e/m obtained by Thomson's cycloid method indicate that the smallest of the positive ions present possess a mass comparable with that the hydrogen atom. Many of the results obtained, in particular (*a*) those with varying pressures and constant temperatures, (*b*) those at atmospheric pressure in which after removal of all free ions by a field sufficient to produce a saturation current, a current of equal values for ions of both signs was found at an electrode placed behind that on which the saturation voltage acted, and (*c*) the loss of charge of a Faraday cylinder when screened from the action of free ions, indicate that one of the products due to heating aluminium phosphate is in the form of neutral pairs or doublets which afterwards split up into negative and positive ions.

XXIX. *The Convection of Heat from a Body cooled by a Stream of Fluid. By* ALEXANDER RUSSELL, *M.A., D.Sc., M.I.E.E., Principal of Faraday House*.*

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1. *Introduction.*

THE phenomenon of the convection of heat at the surface of a body immersed in a cooling fluid is one which does not lend itself readily to mathematical calculation. If the fluid be a gas the variations of the pressure, density, and velocity at different points of the gas so complicate the problem that little progress towards a complete solution has yet been made. In the case of liquids flowing past a body with appreciable but not excessive velocity, Boussinesq † has found some approximate solutions which deserve to be more widely known. The author has therefore thought that it would be useful to give the proofs in full of the more practical of Boussinesq's formulæ, laying stress on their limitations, and pointing out some of their applications. The author also discusses the important problem of the heating, due to stream-line convection, of a liquid flowing through a cylindrical tube, and gives a table by means of which approximate solutions can be found without much difficulty.

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* Read July 8, 1910.

† *Théorie Analytique de la Chaleur,* t. ii. 1903, and *Journal de Mathématiques,* 6ᵉ Série, t. i. (1905).

2. *Historical.*

The differential equations the solutions of which would give the flow of heat through a fluid were first given by Fourier [*]. The fundamental equation was put into a more manageable form by Poisson [†], but neither he nor Fourier gave any solution of it.

Poisson writes the equation as follows :—

$$c\frac{\mathrm{D}\theta}{\mathrm{D}t}=\frac{\partial}{\partial x}\left(k\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial\theta}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial\theta}{\partial z}\right), \quad . \quad (1)$$

where θ is the temperature of the fluid at the point (x, y, z), c the capacity for heat per unit volume, k the conductivity, and $\mathrm{D}\theta/\mathrm{D}t$ the rate at which the temperature of a particle of fluid passing through the point (x, y, z) is increasing in the direction of the motion of the fluid at the point. When written in this form it is interesting to notice how similar this equation is to the equation of the flow of heat through a solid body. We may also write

$$\frac{\mathrm{D}\theta}{\mathrm{D}t}=\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+w\frac{\partial\theta}{\partial z}, \quad . \quad . \quad . \quad (2)$$

where u, v, and w are the component velocities of the current at the point (x, y, z) parallel to the three axes respectively.

In addition to equation (1) we have the ordinary hydrodynamical equations [‡], namely, the equation of continuity and the three equations of Euler.

A. Oberbeck [§] discusses the general equations, and gives a solution for a special case. In a valuable paper [‖] L. Lorenz obtains an approximate solution for the case of a heated strip cooling in air. When the strip is protected from draughts he proves that the heat convected from it varies as $\theta^{5/4}$ where θ is the difference of temperature between the strip and the air before it is heated by the strip. L. Graetz * finds the mathematical equation for thermal conduction in a liquid flowing through a cylindrical tube, and obtains a solution in terms of Bessel's functions. Harold Wilson † gives a solution of this problem, taking the viscosity of the liquid into account; but unfortunately his solution is only applicable to a very special case.

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[*] *Mémoires de l'Académie*, t. xii. p. 507 (1820), or *Œuvres de Fourier* (Darboux's Edition), t. ii. p. 275.

[†] *Théorie Mathématique de la Chaleur*, chapter iv. (1835).

[‡] Lamb's 'Hydrodynamics,' chap. i.

[§] *Ann. der Physik*, vii. p. 271 (1879).

[‖] *Ann. der Physik*, xiii. p. 582 (1881).

Boussinesq was the first to state clearly the laws for the cooling of a heated body by a stream of liquid when the flow is not turbulent. In 1901 ‡ he published the formula for the cooling of a strip by a liquid flowing past it in a direction at right angles to its length and parallel to its breadth. Four years afterwards (*l. c. ante*) the same author published the solution of the problem of the convection of heat from a horizontal cylindrical rod of elliptical cross-section immersed in a liquid flowing in a direction at right angles to the axis of the rod. He also gave the solution for the similar problem of the convection of heat from an ellipsoidal shaped body.

3. *The Assumptions made.*

In order to simplify the mathematical work the following assumptions are made. The liquid is supposed to be athermanous, that is, opaque to heat rays. It is also supposed to have no viscosity. The liquid therefore slips past the surface of the solid. In addition it is supposed to be incompressible. Hence we should only expect the solutions to give roughly approximate values when applied to the problem of spheres and cylinders being cooled by currents of air. It is instructive to notice, however, that Boussinesq's result, that the convection of heat by a stream of liquid from a sphere or a cylinder maintained at a constant temperature varies as the difference of temperature between the solid and the liquid and as the square root of the velocity of the current, is in good agreement with the results obtained by P. Compan [*] from experiments with spheres in draughts of air, and also with Kennelly's [†] results for the cooling of cylindrical wires. Boussinesq's theoretical results also would lead us to expect that the loss of heat per square centimetre of the surface of a wire would be greater the smaller the diameter of the wire. This is in agreement with the experiments of Cardani [‡], Ayrton and Kilgour [§], Sala [‖], and Kennelly [¶].

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* *Ann. der Physik*, xviii. p. 79 (1883).

† Camb. Phil. Soc. Proceedings, xii. p. 406 (1904).

‡ *Comptes Rendus*, cxxxiii. p. 257.

We also show later on that in certain cases the fusing-current of wires when immersed in a stream of liquid varies as the 1·25th power of the radius of the wire. This agrees with experimental results obtained by Schwartz and James [**] for wires in air.

The further assumptions are made that the thermal conductivity of the liquid is very small and that the variation in its density does not appreciably alter the shape of the trajectories of the liquid particles in the immediate neighbourhood of the solid from the shape they have during isothermal flow. The former assumption is true in most practical cases, and the latter is permissible when the velocity of the current is appreciable and no eddies are formed.

It is interesting to remember that in Hele-Shaw's [††] method of reproducing the stream-lines of a perfect fluid flowing past an obstacle in two dimensions, a thin film of viscous liquid, glycerine for example, is employed, and results of high accuracy are obtained. Even for a thick film, the shape of the lines does not alter much from the ideal case, and hence the assumption that the stream-lines coincide with the stream-lines of a perfect fluid is not a serious one.

The surface of the solid being cooled by the current is supposed to be isothermal, and the liquid in immediate contact with it at any instant is supposed to have the same temperature as the solid. These two assumptions are quite legitimate.

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[*] *Ann. de Chim. et Phys.* xxvi. p. 488 (1902).

[†] Amer. Inst. Elect. Engin. Proc. July 1909.

[‡] *Nuov. Cim.* xxx. p. 33 (1891).

[§] Phil. Trans. clxxxiii. part i. p. 371 (1892).

[‖] *Nuov. Cim.* iv. p. 81 (1896).

[¶] *L. c. ante.*

[**] Journ. Inst. Elect. Engin. xxxv. p. 364 (1905).

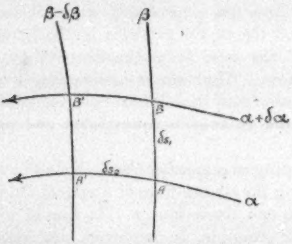
[††] Brit. Assoc. Report, 1898. In this report Sir G. G. Stokes gives a theoretical proof of the method.

4. *Flow in Two Dimensions.*

Making the above assumptions we shall now obtain the differential equation which determines the temperature at any point of the liquid, once the steady state has been established.

Let us suppose that the velocity of the liquid at a great distance from the solid being cooled is V. In our problem it is convenient to denote the hydrodynamical stream-function by $V\alpha$, and the velocity-potential by $V\beta$. In fig. 1, AA′ and BB′ denote adjacent stream-lines, and A′B′ and AB denote adjacent equipotential curves. Let

Fig. 1.



$$AA' = \partial s_2,$$

and let

$$AB = \partial s_1.$$

Then, by hydrodynamics,

$$\frac{\partial V\alpha}{\partial s_1} = \frac{\partial V\beta}{\partial s_2} = q, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where q is the velocity of the flow of the liquid at the point (s_1, s_2).

Consider the flow of heat in the time Dt into a prism having unit length, and having ABB′A′ for its cross-section (fig. 1). We suppose that this prism is moving with the liquid, and its velocity is therefore q. Let θ be the initial temperature of the liquid inside this prism, and let $\theta + \mathrm{D}\theta$ be the temperature at the time $\mathrm{D}t$. The gain by the flow of heat from AA′ to BB′ during this interval is

$$\frac{\partial}{\partial s_1}\left(k\frac{\partial \theta}{\partial s_1}\right)\partial s_1\,\partial s_2\,\mathrm{D}t,$$

and the gain by the flow from AB to A′B′ is

$$\frac{\partial}{\partial s_2}\left(k\frac{\partial \theta}{\partial s_2}\right)\partial s_1\,\partial s_2\,\mathrm{D}t,$$

where k is the conductivity of the liquid. Hence, if c denote the capacity for heat of the liquid per unit volume, the gain of heat by the element contained by the prism is $c\,\partial s_1\,\partial s_2\,\mathrm{D}\theta$. We have, therefore,

$$c\,\partial s_1\,\partial s_2\,\mathrm{D}\theta = \left\{\frac{\partial}{\partial s_1}\left(k\frac{\partial \theta}{\partial s_1}\right) + \frac{\partial}{\partial s_2}\left(k\frac{\partial \theta}{\partial s_2}\right)\right\}\partial s_1\,\partial s_2\,\mathrm{D}t,$$

and thus

$$c\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\partial}{\partial s_1}\left(k\frac{\partial \theta}{\partial s_1}\right) + \frac{\partial}{\partial s_2}\left(k\frac{\partial \theta}{\partial s_2}\right). \quad . \quad . \quad . \quad (4)$$

Since we suppose that the liquid is flowing in the direction AA′, we have, when the steady state is reached,

$$c\frac{\mathrm{D}\theta}{\mathrm{D}t} = c\frac{\partial \theta}{\partial s_2}\cdot\frac{\partial s_2}{\partial t} = -cq\frac{\partial \theta}{\partial s_2},$$

and hence, assuming k constant, we get

$$-\frac{\partial \theta}{\partial s_2} = \frac{k}{cq}\left\{\frac{\partial^2 \theta}{\partial s_1^2} + \frac{\partial^2 \theta}{\partial s_2^2}\right\}. \quad . \quad . \quad . \quad . \quad (5)$$

As q varies with both s_1 and s_2, it appears at first sight as if it would be very difficult to obtain a solution of this equation. If, however, we alter the variables from s_1 and s_2 to α and β, the equation simplifies in a remarkable way.

We have

$$\frac{\partial \theta}{\partial s_1} = \frac{\partial \theta}{\partial \alpha}\frac{\partial \alpha}{\partial s_1} + \frac{\partial \theta}{\partial \beta}\frac{\partial \beta}{\partial s_1},$$

and thus

$$\frac{\partial^2 \theta}{\partial s_1^2} = \frac{\partial^2 \theta}{\partial \alpha^2}\left(\frac{\partial \alpha}{\partial s_1}\right)^2 + 2\frac{\partial^2 \theta}{\partial \alpha \partial \beta}\frac{\partial \alpha}{\partial s_1}\frac{\partial \beta}{\partial s_1} + \frac{\partial^2 \theta}{\partial \beta^2}\left(\frac{\partial \beta}{\partial s_1}\right)^2$$
$$+ \frac{\partial \theta}{\partial \alpha}\frac{\partial^2 \alpha}{\partial s_1^2} + \frac{\partial \theta}{\partial \beta}\frac{\partial^2 \beta}{\partial s_1^2}.$$

A similar equation holds for $\frac{\partial^2\theta}{\partial s_2{}^2}$.

Noticing that

$$\frac{\partial\alpha}{\partial s_1}=\frac{\partial\beta}{\partial s_2}=\frac{q}{V},\quad \frac{\partial\alpha}{\partial s_2}=-\frac{\partial\beta}{\partial s_1}=0,$$

$$\nabla^2\alpha=0,\quad\text{and}\quad \nabla^2\beta=0,$$

we get

$$\frac{\partial^2\theta}{\partial s_1{}^2}+\frac{\partial^2\theta}{\partial s_2{}^2}=\frac{q^2}{V^2}\left(\frac{\partial^2\theta}{\partial\alpha^2}+\frac{\partial^2\theta}{\partial\beta^2}\right).$$

We also have

$$\frac{\partial\theta}{\partial s_2}=\frac{\partial\theta}{\partial\beta}\frac{q}{V}.$$

Hence, substituting these values in (5), we get

$$-\frac{\partial\theta}{\partial\beta}=\frac{k}{cV}\left(\frac{\partial^2\theta}{\partial\alpha^2}+\frac{\partial^2\theta}{\partial\beta^2}\right), \quad . \quad . \quad . \quad . \quad (6)$$

which is a much simpler equation than (5) as the coefficient of the right-hand side is a constant quantity.

Our assumptions allow us to simplify this equation still further. Since the liquid is a very bad conductor of heat, θ alters very rapidly with α but very slowly with β. The term $\partial^2\theta/\partial\beta^2$ is also negligibly small compared with $\partial^2\theta/\partial\alpha^2$. We thus obtain the equation

$$-\frac{\partial\theta}{\partial\beta}=\frac{k}{cV}\frac{\partial^2\theta}{\partial\alpha^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the solution of which has been put into various forms by Fourier and others.

5. *Circular Cylinder.*

We shall now consider the problem of the cooling of a circular cylinder immersed in a stream of liquid with its axis horizontal and at right angles to the direction of flow.

Let us take the origin of co-ordinates on the axis of the cylinder, and let us suppose that the liquid is flowing with velocity V in the direction XO, and that its temperature is zero before it meets the cylinder.

In this case we know* that

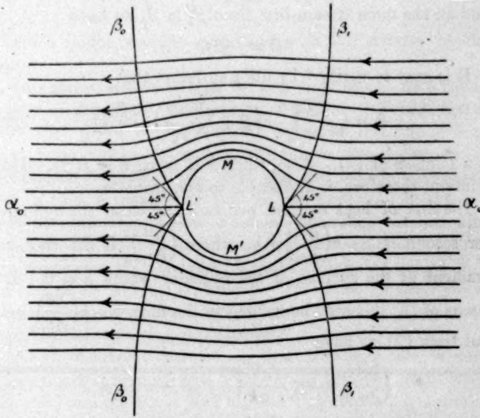
$$\left.\begin{aligned} \alpha &= y - \frac{a^2 y}{r^2} \\ \text{and} \qquad \beta &= x + \frac{a^2 x}{r^2} \end{aligned}\right\}, \quad . \; . \; . \; . \; . \; . \; . \qquad (8)$$

where a is the radius of the cylinder and

$$r^2 = x^2 + y^2.$$

The equation to the stream-line α_0, which flows on the

Fig. 2



Stream lines of Fluid flowing past cylinder.
β_0 & β_1 are the two singular equipotential curves.

surface of the cylinder (fig. 2), is $\alpha = 0$, and the equations to the two equipotential lines, β_0 and β_1, are

$$x\left(1 + \frac{a^2}{r^2}\right) = -2a \quad \text{and} \quad x\left(1 + \frac{a^2}{r^2}\right) = 2a$$

respectively. These curves cut the cylinder and the stream-line α_0 at angles of 45°. We see therefore that the velocity of the liquid at L and L′ must be zero. The velocity at M and M′ is 2V. At a great distance away from the cylinder β_0 and β_1 practically coincide with the lines

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* Lamb's 'Hydrodynamics,' Third Edition, p. 74.

$$x = -2a \quad \text{and} \quad x = 2a.$$

Let us suppose that the temperature flow has become steady and that the temperature of all points on the streamline α_0 is $f(\beta)$. On this stream-line (fig. 2), from $\beta = \infty$ to $\beta = \beta_1$, we have

$$\theta = f(\beta) = 0,$$

and on the same stream-line from β_1 to β_0 we have

$$\theta = f(\beta).$$

It is easy to verify * by differentiation that

$$\theta = \sqrt{\frac{2}{\pi}} \int_0^\infty f\left(\beta + \frac{cV}{k}\frac{\alpha^2}{2\xi^2}\right) \epsilon^{-\frac{\xi^2}{2}} \partial\xi \quad . \quad . \quad (9)$$

is a solution of (7). Also, when α is zero, $\theta = f(\beta)$. This solution, therefore, is applicable to our problem.

The flux of heat emitted per unit length of the cylinder per second is $\Sigma - k\left(\frac{\partial\theta}{\partial s_1}\right)_0 \partial s_2$, where $\left(\frac{\partial\theta}{\partial s_1}\right)_0$ is the thermal gradient at the surface of the cylinder where $\alpha = 0$. By means of (3) this may be written in the form $\Sigma - k\left(\frac{\partial\theta}{\partial\alpha}\right)_0 \partial\beta$. But from (9) we have

$$\left(\frac{\partial\theta}{\partial\alpha}\right)_0 = 2\sqrt{\frac{cV}{\pi k}} \int_0^\infty f'(\beta + \eta^2)\, \partial\eta,$$

where we have written η^2 for $cV\alpha^2/(2k\xi^2)$.

Hence the total flux H of heat per second from unit length of the cylinder is given by

$$H = -4\sqrt{\frac{ckV}{\pi}} \int_{\beta_0}^{\beta_1} \int_0^\infty f'(\beta + \eta^2)\, \partial\beta\, \partial\eta$$

$$= 4\sqrt{\frac{ckV}{\pi}} \int_0^\infty \{f(\beta_0 + \eta^2) - f(\beta_1 + \eta^2)\} \partial\eta. \quad (10)$$

Now $f(\beta_1 + \eta^2)$ is zero from η almost equal to nothing up

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* *Cf.* Boussinesq, *Application des Potentials*, p. 360 (1885).

to η equal to infinity; and $f(\beta_0+\eta^2)$ is θ_0 from η equal to zero up to η equal to $\sqrt{\beta_1-\beta_0}$, and practically vanishes for all greater values of η. Hence

$$\begin{aligned} H &= 4\sqrt{\frac{ckV}{\pi}}\int_0^{\sqrt{\beta_1-\beta_0}} f(\beta_0+\eta^2)\,\partial\eta \\ &= 4\sqrt{\frac{ckV}{\pi}}\sqrt{\beta_1-\beta_0}\,\theta_0 \\ &= 4\sqrt{\frac{s\sigma kV}{\pi}}\sqrt{\beta_1-\beta_0}\,\theta_0, \quad . \ . \ . \ . \ (11) \end{aligned}$$

where s is the specific heat, and σ is the density of the liquid.

This result, which is true for two-dimensional flow round a solid of any shape immersed in a stream of liquid, agrees with that given by Boussinesq. It shows that the loss of heat from the solid is proportional to the difference of temperature between the solid and the liquid. Newton's law is thus verified when the cooling fluid is a liquid. It will be remembered that Newton enunciated his law with reference to the convection and not the radiation of heat. He considered the case of a block of iron being cooled in a current of air flowing uniformly. He states * "*aeris partes aequales aequalibus temporibus calefactae sunt & calorem conceperunt calori ferri proportionalem.*"

A. C. Mitchell † has shown that Newton's law is very approximately true up to a difference of temperature between the solid and the air of 200° C., and P. Compan ‡ has proved it true for temperatures up to 300° C.

In several practical applications the assumption of Newton's law for the convection of heat by fluids leads to results which are found to be in close accordance with experiment. For instance, in the theory of the Irwin § hot-wire oscillograph, the assumption is made that the convection of heat from the heated metal strips which are immersed in convection-currents of oil is proportional to the difference of temperature between the metal and the oil. The very satisfactory results obtained in practice prove that the assumption is approximately correct. It will be seen from § 11 below that, even in the case of the turbulent motion of water through a pipe, Newton's law is very approximately true. It is not applicable, however, to natural free convection from a heated body in a gas or a liquid. In this case* Lorenz's law (see above) is applicable.

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* "Scala Graduum Caloris," Phil. Trans. p. 828, April 1701. The paper is not signed. In *Newtoni Opera Horsleii*, vol. iv. p. 403 (1782), the title is given as "Tabula Quantitatum et Graduum Caloris."

† Roy. Soc. Edin. Trans. xl. 1, p. 39 (1899). ‡ *L. c. ante.*

§ Journ. Inst. of Elect. Engin. vol. xxxix. p. 617 (1907).

In the case of the circular cylinder we find, from the values of β_1 and β_0 given above, that

$$\beta_1 - \beta_0 = 4a,$$

and hence

$$H = 8\sqrt{\frac{s\sigma k V a}{\pi}}\,\theta_0 . \quad . \quad . \quad . \quad . \qquad (12)$$

If we denote the surface of unit length of the cylinder by S, so that $S = 2\pi a$, the expression for the rate at which heat is lost by the cylinder per unit length is generally assumed by engineers to be equal to $hS\theta$, where h is independent of the radius of the cylinder. We see from (12) that the value of h for the perfect liquid is given by

$$h = \frac{4}{\pi}\sqrt{\frac{s\sigma k V}{\pi a}} . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Thus h varies inversely as the square root of the radius of the cylinder, and therefore the assumption that it is constant is not permissible. For example, if the radius of one wire is a hundred times that of another, the average heat per square centimetre of surface which is carried off per second by the liquid from the small wire is ten times greater than from the large wire. We also see that if we quadruple the velocity of the flow of the liquid, the temperature of the wire being maintained constant, the convection of heat is doubled, and if the convection of heat is constant the difference of temperature between the wire and the liquid is halved.

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* P. Compan, *l. c. ante*; H. Ebeling, *Ann. der Physik*, xxvii. 2, p. 391 (1908).

6. *Cylinder with Elliptic Section.*

Let the direction of the current be at right angles to the axis of the cylinder, and let it make an angle α with the major axis of the elliptic section which we take as the axis of X. If

$$l=\cos\alpha \text{ and } m=\sin\alpha,$$

we easily find from the formulæ given in Lamb's 'Hydrodynamics' (p. 70, 3rd ed.) that

$$\beta=lb\sqrt{\frac{a+b}{a-b}}\,\epsilon^{-\eta}\sin\xi+lx$$

$$+ma\sqrt{\frac{a+b}{a-b}}\,\epsilon^{-\eta}\cos\xi+my,$$

where

$$x=c\sin\xi\cosh\eta,\quad y=c\cos\xi\sinh\eta,\quad\text{and}\quad c=\sqrt{a^2-b^2}.$$

On the surface of the cylinder we have

$$a=c\cosh\eta,\quad b=c\sinh\eta,$$

and thus at these points

$$x=a\sin\xi,\quad\text{and}\quad y=b\cos\xi.$$

At points, therefore, on the surface of the cylinder,

$$\beta=l\frac{b}{a}x+lx+m\frac{a}{b}y+my,$$

$$=\left(l\frac{x}{a}+m\frac{y}{b}\right)(a+b),$$

$$=(a+b)\left\{(l^2+m^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\left(\frac{ly}{b}-\frac{mx}{a}\right)^2\right\}^{\frac{1}{2}}$$

$$=(a+b)\left\{1-\left(\frac{ly}{b}-\frac{mx}{a}\right)^2\right\}^{\frac{1}{2}}.$$

Thus β has extreme values when

$$x=la,\quad y=mb,$$

and when

$$x=-la\quad\text{and}\quad y=-mb,$$

Hence we find that

$$\beta_1-\beta_0=2(a+b).$$

Substituting this value in (11) we get

$$H=4\sqrt{\frac{s\sigma k V}{\pi}}\sqrt{2(a+b)}\,\theta_0. \quad . \quad . \quad . \quad . \quad (14)$$

Hence the cooling of the cylinder by the stream of liquid is independent of the direction in which the stream impinges on it. For a given area of cross-section and a given temperature the cooling power increases with the eccentricity of the ellipse, being a minimum for a cylinder having a circular cross-section. It is not permissible to apply (14) when the liquid is flowing parallel to the minor axis and b/a is a very small quantity. In this case, the velocity of the liquid round the pointed ends of the ellipse would be very high and eddy currents would be formed

7. *Flat Strip.*

If the solid be a thin strip of metal placed so that its length is perpendicular and its surface parallel to the direction of flow, we have by (11)

$$H=4\sqrt{\frac{s\sigma k V b}{\pi}}\,\theta_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where b is the breadth of the strip. This also follows from (14). The convection of heat h per square centimetre of the surface of the strip per degree of temperature per second is given by

$$h=2\sqrt{\frac{s\sigma k V}{\pi b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If the strip be bent so as to form a hollow cylinder of circumference b, we have, by (13)

$$h=4\sqrt{\frac{2}{\pi}}\sqrt{\frac{s\sigma k V}{\pi b}}$$

$$=3{\cdot}19\sqrt{\frac{s\sigma k V}{b\pi}} \text{ approximately}$$

Hence the average convection per square centimetre of effective surface is considerably increased.

8. *Cylindrical Tube.*

Let the length of the tube be b, its temperature θ_0 and the velocity of the liquid flowing through it V.

In this case, when the steady state is attained, Poisson's equation (1) gives us

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial x^2}=\frac{s\sigma V}{k}\frac{\partial\theta}{\partial x}.$$

As the mathematical formulæ are complex we shall simplify the work by neglecting the conduction of heat in the direction of the flow, in which case

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}=\frac{s\sigma V}{k}\frac{\partial\theta}{\partial x} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is easy to show that the equation *

$$y=\frac{4\theta_0}{\pi}\sum_0^\infty\frac{1}{2n+1}\sin\frac{(2n+1)\pi l}{2a}\cos\frac{(2n+1)\pi x}{a} \quad . \quad (17)$$

makes y, θ_0 from $x=-l/2$ to $+l/2$, 0 from $l/2$ to $a-l/2$, $-\theta_0$ from $a-l/2$ to $a+l/2$, 0 from $a+l/2$ to $a+3l/2$, and so on periodically.

Let us now consider an infinitely long tube. Take the origin at the centre of a portion of it of length l which is maintained at temperature θ_0. Let the contiguous portions be of length $a-l$ and be kept at zero temperature, and let the portions beyond these be of length l and be at temperature $-\theta_0$, and so on. Then by taking a/l sufficiently great we can ensure that the liquid entering the hot portion of the tube is practically at zero temperature.

Writing equation (16) in the form

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}=\frac{(2n+1)m^2}{(2n+1)(\pi/a)}\frac{\partial\theta}{\partial x}, \quad . \quad . \quad . \quad (18)$$

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* Russell, 'Alternating Currents,' vol. ii. p. 388.

where $m^2=\pi s\sigma V/ak$, we deduce that *

$$\begin{aligned}\theta=\frac{4\theta_0}{\pi}\Bigg\{&\frac{\operatorname{ber} mR \operatorname{ber} mr+\operatorname{bei} mR \operatorname{bei} mr}{\operatorname{ber}^2 mR+\operatorname{bei}^2 mR}\frac{\pi l}{2a}\cos\sin\frac{\pi x}{a},\\
&+\frac{1}{3}\frac{\operatorname{ber} m\sqrt{3}R \operatorname{ber} m\sqrt{3}r+\operatorname{bei} m\sqrt{3}R \operatorname{bei} m\sqrt{3}r}{\operatorname{ber}^2 m\sqrt{3}R+\operatorname{bei}^2 m\sqrt{3}R}\sin\frac{3\pi l}{2a}\cos\frac{3\pi x}{a},\\
&+\ldots\ldots\\
&-\frac{\operatorname{ber} mR \operatorname{bei} mr-\operatorname{bei} mR \operatorname{ber} mr}{\operatorname{ber}^2 mR+\operatorname{bei}^2 mR}\sin\frac{\pi l}{2a}\sin\frac{\pi x}{a},\\
&-\frac{1}{3}\frac{\operatorname{ber} m\sqrt{3}R \operatorname{bei} m\sqrt{3}r-\operatorname{bei} m\sqrt{3}R \operatorname{ber} m\sqrt{3}r}{\operatorname{ber}^2 m\sqrt{3}R+\operatorname{bei}^2 m\sqrt{3}R}\sin\frac{3\pi l}{2a}\sin\frac{3\pi x}{a},\\
&-\ldots\ldots\Bigg\}\end{aligned}$$

where R is the radius of the tube.

This value of θ satisfies (18), and when $r=R$, $\theta=\theta_0$ from $-l/2$ to $+l/2$, &c., and thus the boundary conditions are satisfied.

Hence the loss of heat H per second from the portion of the tube from $-l/2$ to $+l/2$ is given by

$$\begin{aligned}H&=\int_{-l/2}^{+l/2}2\pi Rk\frac{\partial\theta}{\partial r}\,dx,\text{ when } r=R,\\
&=16R\sqrt{\frac{s\sigma kVa}{\pi}}\,\theta_0\Bigg\{f(mR)\sin^2\frac{\pi l}{2a},\\
&\qquad+\frac{1}{3\sqrt{3}}f(m\sqrt{3}R)\sin^2\frac{3\pi l}{2a},\\
&\qquad+\ \ldots\ldots\Bigg\}\quad\ldots\ldots\ (19)\end{aligned}$$

where $f(\xi)=\frac{Z}{X}$,

$$=\frac{\operatorname{ber}\xi\operatorname{ber}'\xi+\operatorname{bei}\xi\operatorname{bei}'\xi}{\operatorname{ber}^2\xi+\operatorname{bei}^2\xi}.$$

It will be seen that H is proportional to θ_0, but except in the case when mR is great it is not proportional to

$$R\sqrt{s\sigma kVl}.$$

I have to thank Mr. H. Savidge for permission to publish

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* Russell, Phil. Mag. April 1909, p. 535.

the table of the values of Z/X given below. This, in addition to the table * and formulæ † for Z/X previously published, makes (19) a practical formula.

9. *Tables of the Values of the Function* Z/X.

ξ.	Z/X.	ξ.	Z/X.	ξ.	Z/X.
3·0	0·5399	4·4	0·5925	5·8	0·6180
3·2	0·5550	4·6	0·5964	6·0	0·6211
3·4	0·5656	4·8	0·6002	6·2	0·6240
3·6	0·5734	5·0	0·6040	6·4	0·6267
3·8	0·5793	5·2	0·6077	6·6	0·6293
4·0	0·5842	5·4	0·6113	6·8	0·6317
4·2	0·5885	5·6	0·6147	7·0	0·6339

10. *Simplified Formulæ for Cylindrical Tube.*

In many practical cases $m^2\mathrm{R}^2$, which equals $\pi s\sigma \mathrm{VR}^2/ak$, is a large number, and thus we may write

$$f(\xi)=\frac{1}{\sqrt{2}}-\frac{1}{2\xi}.$$

When we do this (19) becomes

$$\mathrm{H}=8\sqrt{2}\mathrm{R}\sqrt{\frac{s\sigma k\mathrm{V}a}{\pi}}\,\theta_0\left\{\sin^2\frac{\pi l}{2a}+\frac{1}{3\sqrt{3}}\sin^2\frac{3\pi l}{2a}+\frac{1}{5\sqrt{5}}\sin^2\frac{5\pi l}{2a}+\ldots\right\}$$
$$-\frac{8ak}{\pi}\theta_0\left\{\sin^2\frac{\pi l}{2a}+\frac{1}{3^2}\sin^2\frac{3\pi l}{2a}+\ldots\right\}.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(20)$$

If $a=2l$, so that we have lengths of the tube l, at temperatures θ_0 and $-\theta_0$, separated by lengths l at temperature zero, we have

$$\mathrm{H}=4\sqrt{2}\mathrm{R}\sqrt{\frac{s\sigma k\mathrm{V}a}{\pi}}\,\theta_0\left(1+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+\ldots\right)-\frac{\pi}{2}ak\theta_0,$$

and noticing that $\sum\limits_0^\infty\frac{1}{(2n+1)^{3/2}}=1{\cdot}6888$ approximately, we get

$$\mathrm{H}=13{\cdot}51\mathrm{R}\sqrt{\frac{s\sigma k\mathrm{V}l}{\pi}}\,\theta_0-\pi lk\theta_0\,.\quad.\quad.\quad.\quad(21)$$

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* Savidge, Phil. Mag. Jan. 1910, p. 56.

† Russell, Phil. Mag. April 1909, pp. 529 & 532.

When the last term can be neglected this becomes

$$H = 13{\cdot}51 R \sqrt{\frac{s\sigma k V l}{\pi}}\, \theta_0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Similarly when $a = 3l$ we get

$$H = 2\sqrt{6} R \sqrt{\frac{s\sigma k V l}{\pi}} \left(1 + \frac{1}{\sqrt{3}}\right) \sum_0^\infty \frac{1}{(2n+1)^{3/2}}$$

$$= 13{\cdot}05 R \sqrt{\frac{s\sigma k V l}{\pi}}\, \theta_0 , \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and when $a = 6l$,

$$H = 12{\cdot}86 R \sqrt{\frac{s\sigma k V l}{\pi}}\, \theta_0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If the temperature of the liquid entering the tube be zero and the temperature of the liquid leaving it be practically zero, except at points very close to the tube, we may deduce a formula from Boussinesq's formula (15) for a strip as follows :

$$H = \int_0^{2\pi R} 2\sqrt{\frac{s\sigma k V l}{\pi}}\, \theta_0 \partial s,$$

$$= 4\pi R \sqrt{\frac{s\sigma k V l}{\pi}}\, \theta_0,$$

$$= 12{\cdot}57 R \sqrt{\frac{s\sigma k V l}{\pi}}\, \theta_0 \quad . \quad . \quad . \quad . \quad . \quad (25)$$

This result is in good agreement with the preceding three formulæ.

11. *Turbulent Flow.*

It must be carefully noticed that in the above problem we have supposed that the particles of water flow in straight lines parallel to the axis of the tube. It is known, however, that in practice, when the velocity exceeds a certain critical value, the flow becomes turbulent and the eddy currents cause the particles of liquid to flow in sinuous paths. The theory of the convection of heat in this case has been studied by Osborne Reynolds *. He states that it is due to two causes. 1. The natural internal diffusion when at rest.

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* Proc. of the Lit. and Phil. Soc. of Manchester, vol. xiv. p. 9 (1874).

2. The eddies caused by visible motion which mix the fluid up and continually bring fresh particles into contact with the surface. In our notation, the formula deduced is

$$H = A\theta + B\sigma V\theta,$$

where A and B are constants.

As the first term is small, H is approximately proportional to V. T. E. Stanton [*], who has given an experimental verification of Reynolds's theory, finds that H varies as V^n where the value of n is a little less than unity.

E. G. Coker [†] and S. B. Clement have proved that the critical velocity at which stream-line motion changes to eddy motion varies directly as the viscosity of the liquid and inversely as the radius of the tube.

12. *Electric Current required to fuse a Wire.*

Let us suppose that the wire is horizontal with its axis at right angles to the direction of the flow of the liquid in which it is immersed, and let us suppose that the electric current through it is increased very slowly until the wire fuses. Let a be the radius in centimetres of the wire which we suppose to be cylindrical, C the current in amperes, θ the steady temperature corresponding to this current, and ρ_t the volume resistivity of the metal at t° C. When the steady state is attained the heat generated by the current per unit length of the wire per second must equal the heat convected. Hence, by (12),

$$0{\cdot}239C^2 \frac{\rho_\theta}{\pi a^2} = 8\sqrt{\frac{s\sigma k V a}{\pi}}\,\theta \quad . \quad . \quad . \quad . \quad (26)$$

and thus,

$$C = 7{\cdot}70\,(\theta/\rho_\theta)^{1/2}(s\sigma k V)^{1/4}a^{1{\cdot}25} \quad . \quad . \quad . \quad . \quad (27)$$

If θ be the melting temperature of the metal, we see that the fusing current varies as $(s\sigma k V)^{1/4}$, and also as the 1·25th power of the radius of the wire. This latter result is in good agreement with experimental results obtained by Professor Schwartz (*l. c. ante*). In his experiments the wire was stretched horizontally in air. The current through it was then increased very slowly until the wire melted, the reading on the ammeter in the circuit at this instant giving the fusing current. Before it melted a vertical stream of air was flowing past the wire, the heating of the air by the wire causing this convection current. For wires of small diameter this current would be approximately constant, and so making the assumption that the formulæ given above for cooling by incompressible fluids may be applied for gases, we see that the fusing current varies as the 1·25th power of the radius.

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[*] Phil. Trans. vol. 190, p. 67 (1897).

[†] Phil. Trans. vol. 201, p. 45 (1903).

13. *Schwartz's Experimental Results.*

Expressing the fusing current by λa^n where λ and n are constants for a given metal, the following results were obtained for λ and n.

Metal.	Length of fuse.	S.W.G.	Fusing Currents.	λ.	n.
Copper (tinned) .	5 cms. and upwards ...	47 to 33	1 to 10	358	1·20
"	3·8 cms.	"	"	491	1·26
Tin	7·6 cms. and upwards .	43 to 20	"	147	1·13
"	15 cms. "	20 to 7	10 to 80	239	1·32
Silver	12·7 cms. "	35 to 18	7 to 70	967	1·29
Aluminium	10 cms. "	42 to 20	2 to 30	640	1·27

In the case of most of the wires placing them vertically did not affect the value of n. Before this paper was published electricians, making the assumption that the heat emitted per unit surface of the wire was independent of its radius, deduced that n should be 1·5.

14. *Steady Temperature of a Wire carrying an Electric Current.*

If we assume that the volume resistivity of a wire varies with temperature according to the law

$$\rho_\theta = \rho_0 (1 + \alpha\theta),$$

we get by (26),

$$\theta \left\{ 8 \sqrt{\frac{s\sigma k V a}{\pi}} - 0{\cdot}239 C^2 \frac{\rho_0 \alpha}{\pi a^2} \right\} = 0{\cdot}239 C^2 \frac{\rho_0}{\pi a^2} \quad . \ (28)$$

and thus θ can be easily computed. The value of C must of course be less than the fusing current.

Suppose, for example, that the wire is being cooled by a stream of ice-cold water. We shall take

$$s = \sigma = 1 \quad \text{and} \quad k = 0{\cdot}0016.$$

Hence

$$\theta = 1 \Big/ \left\{ 2{\cdot}37 \frac{\sqrt{\mathrm{V}a^5}}{\mathrm{C}^2 \rho_0} - \alpha \right\}. \quad . \quad . \quad . \quad . \quad (29)$$

If the rod were of pure copper

$$\rho_0 = 1{\cdot}56 \times 10^{-6}, \quad \text{and} \quad \alpha = 0{\cdot}004.$$

If, in addition,

$$a = 0{\cdot}25 \text{ cm.} \quad \mathrm{V} = 25 \text{ cm./sec.} \quad \text{and} \quad \mathrm{C} = 1600 \text{ amperes,}$$

we readily find from (29) that θ is $11^\circ{\cdot}3$ C.

15. *The Effect on the Convection of Heat from a Cylinder of putting a Covering round it* [*].

Let a be the radius of the cylinder which we suppose to be maintained at a constant temperature θ_1, and let b be the outer radius of the insulating covering. We shall suppose that k_1, the thermal conductivity of the insulating covering, is large compared with the conductivity k of the cooling liquid, so that we can suppose the outer surface of this covering to be isothermal.

The equation to the steady flow of heat across the insulating covering is

$$-k_1 2\pi r \frac{\partial \theta}{\partial r} = \text{constant} = \mathrm{H},$$

and hence,

$$\theta_1 - \theta_0 = \frac{\mathrm{H}}{2\pi k_1} \log_e \frac{b}{a}. \quad . \quad . \quad . \quad . \quad (30)$$

where θ_0 is the temperature of the outer surface of the covering. By (12), we find that

$$\theta_1 = \frac{\mathrm{H}\sqrt{\pi}}{8\sqrt{s\sigma k \mathrm{V} b}} + \frac{\mathrm{H}}{2\pi k_1} \log_e \frac{b}{a}. \quad . \quad . \quad . \quad (31)$$

Let us now consider how the temperature θ_1 of the wire varies with the thickness $b - a$ of the insulating covering when H remains constant. We have

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[*] *Cf.* L. Roy, *Soc. Int. Elect. Bull.* p. 69 (1910).

$$\frac{\partial \theta_1}{\partial b} = \frac{\mathrm{H}}{2\pi k_1 b^{3/2}} \left\{ b^{1/2} - \frac{\pi \sqrt{\pi k_1}}{8 \sqrt{s\sigma k \mathrm{V}}} \right\}.$$

Hence if a be less than $\pi^3 k_1^2/(64 s\sigma k\mathrm{V})$, we see that when the thickness of the covering is very small $\partial\theta_1/\partial b$ is negative, and thus putting on a thin layer of insulating material will have the effect of lowering the temperature of the wire. When $b = \pi^3 k_1^2/(64 s\sigma k\mathrm{V})$ the temperature of the wire has its minimum value θ_{min} which is given by

$$\theta_{\mathrm{min}} = \frac{\mathrm{H}}{\pi k_1} \left\{ 1 + \log_e \frac{\pi^{3/2} k_1}{8 \sqrt{s\sigma k \mathrm{V} a}} \right\} \quad . \quad . \quad . \quad (32)$$

The following simple experiment illustrates this effect. Portions of a piece of thin manganin wire are insulated with glass, the rest being left bare. When placed in a current of air and heated electrically the bare pieces of wire glow brilliantly, but the portions covered by the glass are quite dark and are therefore at a much lower temperature.

In very high tension systems for the electric transmission of power the overhead wires are sometimes surrounded with coronæ which appreciably increase the transmission losses. The author has previously suggested that the losses would be diminished by insulating the overhead wires with a suitable material of high electric strength. The above analysis indicates that this procedure instead of diminishing the permissible current in the wires would actually, in many cases, allow an appreciably greater current to be transmitted for the same rise of temperature of the wire.

In conclusion, I have to thank Professor Charles Lees, F.R.S., for his kind help in giving me a long list of references to papers on this subject.

ABSTRACT.

Attention is directed to certain deductions made by Boussinesq from the mathematical theory of the conduction of heat in liquids. Complete proofs are given of Boussinesq's formulæ, stress being laid on their limitations, and some of their practical applications are pointed out. It is proved that when a hot body is immersed in a stream of liquid flowing with

constant velocity, the cooling is proportional to the difference of temperature between the body and the liquid. Newton proved experimentally in 1701 that this law was true for the case of a hot body being cooled by a draught of air. He enunciated his law with reference to the forced convection of heat from a body, and not, as is often stated, to the natural free convection from it. Lorenz has shown that in special cases the natural convection of heat will vary as the 1·25th power of the difference of temperature. Provided that the velocity of the cooling draught is kept constant between certain limits, Compan has shown that Newton's law is very approximately true even when the difference of temperature is as high as 300° C. Another deduction from the formulæ proved in the paper is that the cooling is very approximately proportional to the square root of the velocity of the convection current.

The Author gives the solution of the problem of the heating of a liquid flowing steadily, with a velocity less than the critical velocity, through a cylindrical tube which is maintained at constant temperature. It is shown that, in many practical cases, the heating power of the tube varies as $R\theta\sqrt{s\sigma kVl}$, where R is the radius of the tube, θ the difference of temperature between the tube and the liquid, s the specific heat, σ the density, k the conductivity, V the velocity of flow, and l the length of the tube.

It is proved that if a wire be immersed in a stream of liquid with its length at right angles to the direction of flow, the electric current which will fuse the wire varies as the 1·25th power of the diameter of the wire.

Finally, the effect on the cooling of an electrically heated cylinder by a stream of liquid, of putting an insulating wrapping round it, is considered. It is shown that in certain cases the effect of this procedure is to lower the temperature of the cylinder, an effect which can be easily demonstrated experimentally. In order to simplify the mathematical work, only the case of incompressible fluids is considered. Experimental results, however, obtained by various physicists are quoted to show that some of the formulæ are approximately true for the cooling of heated bodies by convection with currents of air.

XXX. *On Hysteresis Loops and Lissajous' Figures, and on the Energy wasted in a Hysteresis Loop.* *By* Professor SILVANUS P. THOMPSON, *D.Sc.*, *F.R.S.* *

[Plate VII.]

§ 1. PROBABLY many physicists have attempted to find an explanation of the forms of the looped curves which express the hysteresis exhibited by iron and steel when subjected to cycles of magnetization. Physical explanations to account for their general shape have indeed been given by Ewing and by Hopkinson. Neither of these pioneers, however, offered any mathematical equations to express their forms; nor, so far as appears, has any other person yet found any, though M. Pierre Weiss has put forward an electronic theory to account for the principal features.

According to Ewing's molecular hypothesis of magnetism, the act of magnetization consists in the orientation into a common direction of the axes of the elementary magnets constituted by the iron molecules which, when the mass of iron is in the unmagnetized state, are miscellaneous in their directions, the molecules being then arranged in groups within which the individuals are so oriented as to satisfy amongst themselves their various polarities in a more or less stable equilibrium. When a small magnetizing force is applied and gradually increased, the individual elementary magnets are at first merely slightly deflected towards the line of the magnetizing force, but still remain in their various groups. With larger magnetizing forces and increased deflexions of individual elements, the groupings, or some of them, become unstable, and break up as instability is reached; the elements of the group then suddenly swinging round into a new configuration more nearly in alignment with the impressed magnetic force. The less stable groups will be first affected, the more stable afterwards, and the most stable will be the last to swing into alignment. When all or nearly all the groups have thus been broken up, any further increase in the magnetizing forces can produce but little effect, though an infinite magnetizing force might be needed to produce absolute alignment of every element. To deduce from this hypothesis an expression for the ascending curve of magnetization, it might be possible to apply the statistical method, under the assumption that the number and variety of the groupings is enormously great. The ratio $d\mathfrak{B}/d\mathfrak{H}$ would represent at each stage the differential permeability of the specimen, or the rate (per unit of magnetizing force) at which the magnetization was proceeding; and this would at every stage be proportional to the probable number of groups that were coming into alignment, and to the magnitude of the aligning force. The expression would thus assume the form

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* Read July 8, 1910.

$$\frac{d\mathfrak{B}}{d\mathfrak{H}} = a \, . \, \epsilon^{-(\mathfrak{H}-b)^2} \, ;$$

where a and b are constants, the one denoting the maximum value of $d\mathfrak{B}/d\mathfrak{H}$, the other the particular value of $\mathfrak{H}$ at which that maximum is reached. The difficulty of integrating this expression is not the only objection to it; for it would at best give only the ascending curve of magnetization, and additional assumptions would need to be made before it could be adapted to express the descending branch.

§ 2. Hysteresis loops, as found by experiment, can however be considered from a wholly different standpoint. Whatever the law connecting $\mathfrak{B}$ and $\mathfrak{H}$, the area enclosed by the loop measures the amount of energy lost in the iron in the cycle of magnetic operations; the value, in ergs per cubic centimetre, being $\frac{1}{4\pi}\int \mathfrak{B} \, . \, d\mathfrak{H}$. Let it be assumed that the variations of $\mathfrak{H}$ are such that $\mathfrak{B}$ passes from the value $+\mathfrak{B}_1$ to $-\mathfrak{B}_1$ and back, through a regular cycle of values according to the expression

$$\mathfrak{B} = \mathfrak{B}_1 \cos \theta \, ;$$

then, in general, the variations of $\mathfrak{H}$, though they go through a cycle, will not be capable of being expressed by any such simple form, otherwise the shape of the hysteresis loop would be simply an ellipse, or, in the limiting case, a straight line.

The fact that the magnetizing current (and therefore the magnetizing force) in choking coils and transformers does not follow a simple sine or cosine function is well known to all electrical engineers. In these instruments, if worked from a source of alternating electromotive force in which the wave-form of the supply is a pure sine-curve, the wave-form of the flux in the core, and therefore of the flux-density $\mathfrak{B}$ in the core, will also be a pure sine-curve having a lag of exactly 90° with respect to the impressed voltage curve. Or if the impressed voltage is expressed by the equation

$$V = V_0 \sin \theta,$$

the flux-density will be given by the equation

$$\mathfrak{B} = - \mathfrak{B}_0 \cos \theta.$$

But the current will not be capable of being expressed in any such simple fashion *. It has also long been known how these curves of magnetizing current are related to the hysteresis loop. If the form of the hysteresis loop is known, then the form of the magnetizing current can be deduced graphically. An example will be found in Kapp's 'Transformers' (1908), fig. 56, p. 106. Assuming that the flux-density follows the equation given above, let it be plotted as a curve, with maximum ordinate equal to that of the hysteresis loop. Consider any point on this curve, or the point on the loop having the same ordinate; then the ordinate at the same instant on the current curve will be (on some scale) proportionate to the abscissa of the same point of the hysteresis loop; whence it follows that the current curve can be constructed, point by point, through the entire cycle. This process is followed here, save that for convenience the hysteresis loop is turned over through a right angle, so that the values of $\mathfrak{B}$ are taken as abscissæ, and those of $\mathfrak{H}$ as ordinates. In fig. 1 the hysteresis loop A B C D E F

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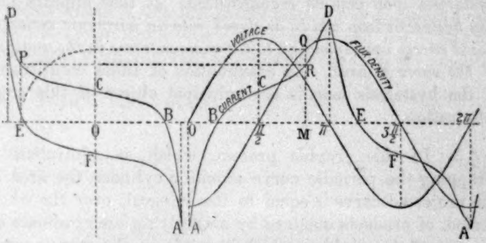
* For a good example of a current curve, see Fleming's 'Alternate Current Transformer,' fig. 186, vol. i. p. 543 (edition of 1896). These irregular current curves seem to have been first observed by Ryan; see 'Transactions of the American Institute of Electrical Engineers,' vol. iii. Jan. 10, 1890. See also the 'Electrician,' vol. xxiv. p. 239, and p. 263, January 1890, and vol. xxv. p. 312, July 25, 1890.

selected, has been taken from Ewing's classical memoir of 1885.

Assuming a closed magnetic circuit, and an impressed voltage of a pure sine form, marked VOLTAGE, the flux-density curve is a sine-curve lagging 90° behind the voltage-curve. Then the current curve is derived by taking any point P on the hysteresis loop, projecting it by first turning its abscissa through a quadrant about the centre O, then tracing along horizontally to the point Q on the flux-density curve, where a vertical line QM is dropped. Then a horizontal line projected from P cuts QM in R, giving thus the corresponding point on the current curve, the peak of which, corresponding to the cusp of the loop, occurs at the time when the flux-density curve is a maximum, and when the voltage curve is at its zero.

Fig. 1.



Relation of Current Curve to Hysteresis.

It will be seen that the wave-form of the current curve reflects, in a certain way, the form of the hysteresis loop. If the loop is sharply cusped, the wave-curve will have corresponding sharp peaks. In fact, the loop consists of the two halves of the wave-curve, folded back one upon another, but with the ordinates differently spaced, exactly as if the wave-curve had been wrapped around a cylinder* and projected upon a plane cutting the cylinder diametrically through the two peaks of the curve.

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* As in the graphic method of harmonic analysis of Clifford described by Perry in Proc. Phys. Soc. vol. xiii.

Now this current curve can be subjected to harmonic analysis, and its harmonic constituents discovered. Each constituent will be a pure sine-curve or cosine curve. If each such constituent be drawn, and then be projected back by reversal of the process by which the wave-curve was obtained, the several constituents will reappear as separate closed curves; and by the summation of these the original hysteresis loop can be reconstituted. It thus appears that *any hysteresis loop can be analysed into an harmonic series of closed curves corresponding to the various terms in the analysis of the current wave.* An examination of these constituents of the hysteresis loop is the principal object of this communication.

§ 3. In this graphic process, which is equivalent to wrapping the periodic curve around a cylinder, the area of the projected curve is equal to the integral, over the whole period, of products obtained by multiplying each ordinate by the sine of the angle at which it stands in the wave-curve; abscissæ in that curve being reckoned as values of angles. (The origin of the cycle is taken where the curve has its negative peak.) In symbols this is equivalent to

$$\int_0^{2\pi} \sin\theta \, . \, \psi(\theta) \, . \, d\theta.$$

Now $\psi(\theta)$, the function which represents the current-curve, may by Fourier's theorem be expanded into the series

$$A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots\ldots A_n \sin n\theta$$
$$+ B_1 \cos\theta + B_3 \cos 3\theta + B_5 \cos 5\theta + \ldots\ldots B_n \cos n\theta.$$

Odd terms only are present, as in all alternating current work; and there is no constant term, because the mean ordinate is already zero.

The constituent terms in the area of the hysteresis loop correspond therefore to the integrated products of $\sin\theta$ into the successive terms of the above series.

§ 4. To investigate the form of the constituent elements of the loop, let us consider a simple harmonic motion $x = X \sin\theta$, where θ stands for $2\pi ft$, f being the fundamental frequency, and X the amplitude. This motion is to be compounded, at right angles, with another simple harmonic motion

$$y = Y_n \sin(n\theta + \phi_n);$$

where Y_n is the amplitude, ϕ_n a possible angle of phase-difference, and n any (odd) integer giving the order of the harmonic. We have then to find an expression for the curve of which x and y are the coordinates. For simplicity we deduce the expressions where $n=1$, $n=3$, and $n=5$, that is for the first, third, and fifth terms of the constituent elements.

FIRST TERM (Fundamental); $n=1$.

We have

$$\frac{x}{X} = \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{y}{Y_1} = \sin(\theta + \phi_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying both sides of (1) by $\cos\phi_1$, we have

$$\frac{x}{X}\cos\phi_1 = \sin\theta \,.\, \cos\phi_1.$$

Also

$$\frac{y}{Y_1} = \sin\theta \,.\, \cos\phi_1 + \cos\theta \,.\, \sin\phi_1.$$

Subtracting this equation from the preceding, we have

$$\frac{x}{X}\cos\phi_1 - \frac{y}{Y_1} = -\cos\theta \,.\, \sin\phi_1. \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying (1) by $\sin\phi_1$, we get

$$\frac{x}{X}\sin\phi_1 = \sin\theta \,.\, \sin\phi_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Squaring (3) and (4), and adding them gives us

$$\left(\frac{x}{X}\right)^2 + \left(\frac{y}{Y_1}\right)^2 - 2\frac{xy}{XY_1}\cos\phi_1 = \sin^2\phi_1. \quad . \quad (5)$$

This is the equation to an ellipse, such as is represented by fig. 2 (Pl. VII.). According to the values given to ϕ_1 there arise three principal cases.

Case (i.). If

$$\phi_1 = \frac{\pi}{2} \quad \text{or} \quad 3\frac{\pi}{2},$$

then

$$\sin\phi_1 = \pm 1 \quad \text{and} \quad \cos\phi_1 = 0,$$

and the equation becomes

$$\frac{x^2}{X^2} + \frac{y^2}{Y_1^2} = 1.$$

This is the equation to an ellipse set orthogonally with respect to the coordinate axes as in fig. 3 (Pl. VII.).

Case (ii.). If

$$\phi_1 = 0, \quad \sin\phi_1 = 0, \quad \text{and} \quad \cos\phi_1 = 1,$$

and the equation becomes

$$\frac{x^2}{X^2} + \frac{y^2}{Y_1^2} - 2\frac{xy}{XY_1} = 0,$$

whence

$$y = +\frac{Y_1}{X}x,$$

which is the equation to a straight line into which the ellipse shrinks as in fig. 4 (Pl. VII.). But its length is limited by the prior expressions, since x and y cannot exceed X and Y_1 respectively.

Case (iii.). If

$$\phi_1 = \pi, \quad \sin\phi_1 = 0, \quad \cos\phi_1 = -1,$$

and the equation becomes

$$\frac{x^2}{X^2} + \frac{y^2}{Y_1^2} + 2\frac{xy}{XY_1} = 0,$$

or

$$y = -\frac{Y_1}{X}x,$$

which is a straight line sloping the reverse way as in fig. 5.

For all other values of ϕ the ellipse takes some intermediate form. The sine-component of the first term in the harmonic analysis of the current curve corresponds to the orthogonal ellipse; the cosine-component to the oblique line form. All the intermediate forms of the ellipse could be obtained by wrapping a sine-curve of period T around a cylinder of diameter T/π and projecting in appropriate directions upon planes parallel to the axis of the cylinder the apparent outline of the sine-curve.

THIRD TERM (Third Harmonic); $n=3$.

Here the two equations are

$$\frac{x}{X} = \sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

$$\frac{y}{Y_3} = \sin(3\theta + \phi_3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad : \qquad (2)$$

$$= \sin 3\theta \,.\, \cos\phi_3 + \cos 3\theta \,.\, \sin\phi_3. \quad . \quad . \qquad (2\,a)$$

But

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$

by known trigonometrical relations.

Inserting for $\sin\theta$ its value from (1), we get

$$\sin 3\theta = \frac{3x}{X} - \frac{4x^3}{X^3}. \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Substituting this value in (2 *a*), we deduce

$$\left(\frac{3x}{X} - \frac{4x^3}{X^3}\right)\cos\phi_3 - \frac{y}{Y_3} = -\cos 3\theta \,.\, \sin\phi_3. \quad . \quad . \qquad (4)$$

Also multiplying (3) by $\sin\phi_3$, we have

$$\left(\frac{3x}{X} - \frac{4x^3}{X^3}\right)\sin\phi_3 = \sin 3\theta \,.\, \sin\phi_3. \quad . \quad . \quad . \qquad (5)$$

Squaring (4) and (5), and adding, we get

$$\left(\frac{3x}{X} - \frac{4x^3}{X^3}\right)^2 + \left(\frac{y}{Y_3}\right)^2 - 2\left(\frac{3x}{X} - \frac{4x^3}{X^3}\right)^2 \frac{y}{Y_3} \,.\, \cos\phi_3 = \sin^2\phi_3. \qquad (6)$$

This is the equation to a figure having the general form of fig. 6 (Pl. VII.), which is indeed the well-known Lissajous' figure, compounded of two vibrations the frequencies of which are as 1 : 3. It could be obtained by taking three complete sine-waves, each of period T/3, and wrapping them around a cylinder of diameter T/π.

Again there arise three cases :—

Case (i.). If $\phi_3=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, then

$$\sin\phi_3=\pm 1 \quad \text{and} \quad \cos\phi_3=0,$$

and then the equation becomes

$$\left(\frac{3x}{X}-\frac{4x^3}{X^3}\right)^2+\left(\frac{y}{Y_3}\right)^2=1.$$

Here the figure is symmetrical with respect to the axes, as in fig. 7. It is, for the third term, what the orthogonal ellipse is for the first term.

Case (ii.). If $\phi_3=0$, then

$$\sin\phi_3=0, \quad \cos\phi_3=1,$$

and the equation reduces to

$$\frac{3x}{X}-\frac{4x^3}{X^3}-\frac{y}{Y_3}=0.$$

Here the trilobate form has shrunk to the form of the curved line (fig. 8) precisely as the ellipse shrank to the oblique line of fig. 4. This line is subject to the limitations that x and y cannot exceed X and Y_3, respectively.

Case (iii.). If $\phi_3=\pi$, then

$$\sin\phi_3=0, \quad \cos\phi_3=-1,$$

and the equation becomes

$$\frac{3x}{X}-\frac{4x^3}{X^3}+\frac{y}{Y_3}=0,$$

the graph of which is fig. 9.

FIFTH TERM (Fifth Harmonic) ; $n=5$.

The two equations now are

$$\frac{x}{X}=\sin\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{y}{Y_5}=\sin(5\theta+\phi_5), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$=\sin 5\theta \,.\, \cos\phi_5+\cos 5\theta \,.\, \sin\phi_5. \quad . \quad . \quad (2a)$$

But

$$\sin 5\theta = 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta.$$

Inserting for $\sin \theta$ its value in (1), we get

$$\sin 5\theta = \frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5} . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this value in (2 a), we deduce

$$\left(\frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5}\right) \cos \phi_5 - \frac{y}{Y_5} = -\cos 5\theta \, . \sin \phi_5 . \quad . \quad (4)$$

Also multiplying (3) by $\sin \phi_5$, we have

$$\left(\frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5}\right) \sin \phi_5 = \sin 5\theta \, . \sin \phi_5 . \quad . \quad (5)$$

Squaring (4) and (5), and adding them, we get

$$\left(\frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5}\right)^2 + \left(\frac{y}{Y_5}\right)^2 - 2\left(\frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5}\right) \frac{y}{Y_5} . \cos \phi_5 = \sin^2 \phi_5 .$$

This is the equation of the general Lissajous' figure of the fifth order, representing the result of compounding two vibrations having relative frequencies of 1 : 5, and a general form like fig. 10 (Pl. VII.).

As before, three cases arise :—

Case (i.). If $\phi_5 = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, then

$$\sin \phi_3 = \pm 1 \quad \text{and} \quad \cos \phi_5 = 0,$$

and then the equation becomes

$$\left(\frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5}\right)^2 + \left(\frac{y}{Y_5}\right)^2 = 1,$$

which is symmetrical as in fig. 11.

Case (ii.). If $\phi_5 = 0$, then

$$\sin \phi_5 = 0, \quad \cos \phi_5 = 1,$$

and the equation reduces to

$$\frac{5x}{X} - \frac{20x^3}{X^3} + \frac{16x^5}{X^5} - \frac{y}{Y_5} = 0,$$

corresponding to fig. 12, subject to limitations as before.

Case (iii.). If $\phi_5=\pi$, then

$$\sin\phi_5=0, \quad \cos\phi_5=-1,$$

and the equation is

$$\frac{5x}{X}-\frac{20x^3}{X^3}+\frac{16x^5}{X^5}+\frac{y}{Y_5}=0,$$

which is the equation to fig. 13.

§ 5. *The Higher Terms. Generalized expression for the Lissajous' curve of the* n*th term.*

The expression for any higher term has the general form

$$G_n^2+\left(\frac{y}{Y_n}\right)^2-2G_n\frac{y}{Y_n}\cdot\cos\phi_n=\sin^2\phi_n,$$

which reduces in the case of symmetry to

$$G_n^2+\left(\frac{y}{Y_n}\right)^2=1,$$

and, in cases where $\phi_n=0$, to

$$G_n\pm\frac{y}{Y_n}=0,$$

in which expressions G_n is written for the series

$$G_n=n\frac{x}{X}-C_3\frac{x^3}{X_3}+C_5\frac{x^5}{X^5}-C_7\frac{x^7}{X^7}+\ \ldots\ +C_n(-1)^{\frac{n-1}{2}}\cdot\frac{x^n}{X^n};$$

where

$$C_r=\frac{n(n^2-1^2)(n^2-3^2)\ \ldots\ldots\ (n^2-\overline{r-2}^2)}{\underline{|r}},$$

where r is an odd number greater than unity.

§ 6. It is now possible to restate the proposition at the end of § 2 in the following way. *Any hysteresis loop can be analysed into an harmonic series of Lissajous' figures* of the kind considered in the §§ 4 and 5.

A number of examples of hysteresis loops were chosen, and subjected to harmonic analysis, to ascertain what constituents were present. The loops chosen relate to various kinds of iron and steel, hard and soft, solid and laminated, taken by various methods; a wide selection being made in order to ascertain the physical significance of the several constituent terms.

In carrying out the analysis the author used the simple approximate method described by him to the Physical Society, Dec. 9, 1904, vol. xix. Proc. Phys. Soc. p. 443, based on an arithmetical process originated by Archibald Smith and generalized by Runge in the *Zeitschrift für Mathematik und Physik*, vol. xlviii. p. 443, 1903. It was found that for the present purpose it sufficed to ascertain the harmonic sine and cosine terms up to the eleventh, and therefore to employ twelve equidistant ordinates in the half-period. The work proceeded on the lines of the simple schedule given by the author on p. 448 of his former paper, with a slight modification to enable the origin of abscissæ to be taken not at the point where the ordinate has zero value, but at that point where the ordinate has its negative maximum. At first the values of the twelve ordinates required for the analysis were taken from the current curve plotted, as explained above in § 2, from the hysteresis loop. But it was seen that it was unnecessary to draw the current curve, and that the values of the ordinates might be taken direct from the hysteresis curve, by taking them not equidistant, but at places corresponding to equidistant points in the axis of abscissæ of the wave-curve, which points, when the curve is wrapped round a cylinder, will no longer appear equidistant.

§ 7. The following are the results :—

Example I. fig. 14, Pl. VII., Ewing's hysteresis loop for pianoforte steel wire, in state of normal temper, being fig. 11, pl. lviii. of Philosophical Transactions, 1885.

The analysis of the values of $\mathfrak{H}$ gives the following coefficients of the harmonics up to the eleventh order :—

Sine Terms.	Cosine Terms.
$A_1 = 32\cdot2$	$B_1 = -45\cdot4$
$A_3 = 7\cdot1$	$B_3 = -20\cdot6$
$A_5 = 0\cdot7$	$B_5 = -10\cdot8$
$A_7 = -0\cdot7$	$B_7 = -5\cdot7$
$A_9 = -0\cdot05$	$B_9 = -4\cdot1$
$A_{11} = -0\cdot06$	$B_{11} = -3\cdot4$

It will be seen that the values of the sine terms beyond A_5 are negligible, and are not greater than the errors due to the approximate nature of the method. The cosine terms are all negative and of decreasing values for the successive orders. In the Plate the wave-curve has been given for comparison, and on it the components A_1, A_3, B_1, B_3, and B_5 have been plotted in dotted wave-curves. For comparison with the hysteresis loop its chief components have also been drawn in dotted lines: the ellipse corresponding to A_1; the trilobate curve of A_3; the oblique straight line of B_1, and the curves for B_3 and B_5.

Example II. fig. 15, Pl. VII., Ewing's loop for annealed iron wire, being fig. 5 of plate lviii. of Phil. Trans. 1885.

The analysis gives:—

$$\begin{array}{ll}
A_1 = 3{\cdot}98 & B_1 = -7{\cdot}38 \\
A_3 = 2{\cdot}14 & B_3 = -4{\cdot}74 \\
A_5 = 1{\cdot}36 & B_5 = -2{\cdot}04 \\
A_7 = 0{\cdot}88 & B_7 = -3{\cdot}78 \\
A_9 = 0{\cdot}16 & B_9 = -2{\cdot}14 \\
A_{11} = 0{\cdot}14 & B_{11} = -1{\cdot}90
\end{array}$$

The ellipse and the straight line, corresponding respectively to A_1 and B_1, have been added in dotted lines in the figure.

Example III. fig. 16, Pl. VII., Ewing's loop for annealed iron wire, being fig. 6, plate lviii. of the same memoir.

The analysis gives:—

$$\begin{array}{ll}
A_1 = 4{\cdot}2 & B_1 = -35{\cdot}4 \\
A_3 = 1{\cdot}8 & B_3 = -25{\cdot}5 \\
A_5 = 0{\cdot}7 & B_5 = -17{\cdot}9 \\
A_7 = 0{\cdot}2 & B_7 = -\ \ 6{\cdot}7 \\
A_9 = 0{\cdot}3 & B_9 = -\ \ 4{\cdot}5 \\
A_{11} = 0{\cdot}2 & B_{11} = -\ \ 0{\cdot}5
\end{array}$$

Example IV. fig. 17, is taken from Lord Rayleigh's paper in the Phil. Mag. xxiii. pp. 225–245, 1887, or Scientific Papers, ii. p. 593, and is the loop obtained with very small magnetizing forces on a specimen of "rather hard Swedish iron."

The analysis gives:—

$$\begin{array}{ll}
A_1 = 0{\cdot}553 & B_1 = -1{\cdot}022 \\
A_3 = 0{\cdot}038 & B_3 = -0{\cdot}094 \\
A_5 = 0{\cdot}006 & B_5 = -0{\cdot}046 \\
A_7 = 0{\cdot}002 & B_7 = -0{\cdot}025 \\
A_9 = 0{\cdot}005 & B_9 = -0{\cdot}023 \\
A_{11} = 0{\cdot}000 & B_{11} = -0{\cdot}012
\end{array}$$

Example V.

The next example is taken from a memoir of K. Ångström in the 'Proceedings' of the Royal Swedish Academy of Sciences, 1899, p. 257, where the curves are given without any scale-values. They relate to a rod of steel containing 0·2 per cent. of carbon. Fig. 18 was observed by a magnetostatic method, fig. 19 by using an alternating current of 20 periods per second, fig. 20 by using one of 60 periods per second.

The analyses, on an arbitrary scale, are :—

Fig. 18.

$A_1 = 0{\cdot}57$	$B_1 = -7{\cdot}29$
$A_3 = 0{\cdot}01$	$B_3 = -0{\cdot}59$
$A_5 = -0{\cdot}02$	$B_5 = -0{\cdot}40$
$A_7 = -0{\cdot}005$	$B_7 = -0{\cdot}23$
$A_9 = -0{\cdot}01$	$B_9 = -0{\cdot}24$
$A_{11} = 0{\cdot}07$	$B_{11} = -0{\cdot}24$

Fig. 19.

$A_1 = 2{\cdot}31$	$B_1 = -8{\cdot}45$
$A_3 = 1{\cdot}08$	$B_3 = -0{\cdot}42$
$A_5 = 0{\cdot}44$	$B_5 = -0{\cdot}04$
$A_7 = 0{\cdot}16$	$B_7 = 0{\cdot}02$
$A_9 = 0{\cdot}05$	$B_9 = -0{\cdot}07$
$A_{11} = 0{\cdot}04$	$B_{11} = -0{\cdot}07$

Fig. 20.

$A_1 = 2{\cdot}74$	$B_1 = -8{\cdot}57$
$A_3 = 1{\cdot}21$	$B_3 = 0{\cdot}16$
$A_5 = 0{\cdot}10$	$B_5 = 0{\cdot}33$
$A_7 = -0{\cdot}09$	$B_7 = 0{\cdot}23$
$A_9 = -0{\cdot}04$	$B_9 = -0{\cdot}01$
$A_{11} = 0{\cdot}09$	$B_{11} = -0{\cdot}03$

The small scale of the original drawings of these three loops makes the values of the higher harmonics quite unreliable. But the comparison is of interest as showing the effect of eddy-currents in the substance of the rod in widening the loops, and in increasing both A_1 and A_3.

§ 8. *Work done in the cycle.*—It was early pointed out by Warburg and by Ewing that the work spent in carrying the iron through a cycle of magnetizing operations was represented by the area of the hysteresis loop. We now consider this from another point of view.

Whatever work is spent in magnetizing the iron is derived from the electric energy which is imparted by the circuit, and this, at any instant, is proportional to the product of the current and potential at that instant. If the voltage is represented by the expression

$$V = V_0 \sin pt,$$

where p stands for $2\pi f$, and if the current, being some periodic function of the time, is represented as

$$C = \psi(pt),$$

then the element of work imparted to the circuit during time dt being $CVdt$, the work given to the iron (if the copper resistance is negligible) during one cycle will be

$$V_0 \int_0^T \sin pt \,.\, \psi(pt) \,.\, dt.$$

But $\psi(pt)$ consists (see § 3) of a series of harmonic sine and cosine terms. The quantities which will be formed by multiplying the members of that series by $\sin pt$, and integrating each product over a whole period, will fall under three kinds, the values of which are known, viz.:—

(i.) $\int_0^T \sin pt \,.\, A_n \sin npt \,.\, dt = 0$, (except when $n = 1$);

(ii.) $\int_0^T \sin pt \,.\, B_n \cos npt \,.\, dt = 0$, (in all cases);

(iii.) $\int_0^T A_1 \,.\, \sin^2 pt \,.\, dt = A_1 \,.\, \frac{T}{2}$.

That is, *the only work done in the cycle* is that done by that constituent of the current which is in phase with the voltage, namely, its fundamental sine-term. *All other constituents are wattless.* And since the area of the loop represents the work done, it follows that the area of the hysteresis loop is equal to the area of the orthogonal ellipse which is its fundamental constituent of the sine series. *The true and fundamental form of every hysteresis loop is therefore an orthogonally placed ellipse.* All departures from that form are wattless—are mere distortions which involve no expenditure of energy.

The area of the hysteresis loop is proportional to the maximum value of $\mathfrak{B}$ and to the amplitude of the first sine term into which the values of $\mathfrak{H}$ (corresponding to the values of the current) can be analysed. If the value of the amplitude of that sine-term be denoted by $\mathfrak{H}_1$, then the area of the loop is accurately given by the expression $\pi \times \mathfrak{B}_{\max.} \times \mathfrak{H}_1$; being the area of the fundamental elliptical constituent. To test this conclusion the areas of the three loops, figs. 18, 19, and 20 were planimetered for comparison with the values of A_1:

	Planimeter reading.	A_1.	Ratio.
Fig. 18	33·8	0·57	594
Fig. 19	140·	2·31	605
Fig. 20	167·5	2·74	610

All the constituent curves belonging to the higher orders have zero areas; the lobes formed by the crossing of their outlines being alternately positive areas and negative areas. This is only another way of saying that the integrals (i.) and (ii.) above are always zero. As for those of form (ii.), they are obviously so, as all cosine constituents shrink up to mere lines.

§ 9. *Presence of Eddy-Currents.*

If the hysteresis loop has been produced by some slow process, absence of eddy-currents may be assumed. But this is by no means the case when alternating currents of ordinary frequencies are used, even if the iron be finely laminated. It therefore remains to be seen how the presence of eddy-currents will affect the size and form of the hysteresis loop. The eddy-current, being a secondary current, will be of pure sinusoidal form only if the inducing electromotive force be of a pure sinusoidal form, and if the resistance and permeability be constant also. But it is not necessarily in phase with the impressed electromotive force, but may lag by magnetic reaction ; and indeed, as is already known, lags by different amounts at different depths below the surface of the iron. Assuming equal permeability and resistance in the different layers, the effect of the eddy-current will be represented with sufficient accuracy by a sine-curve lagging by an amount that will depend on conditions into which there is no need here to enter. For here, again, the only effective component—effective that is in the sense of involving expenditure of energy—is the sine-component in phase with the voltage ; and the element which the sine-component contributes to the loop is an orthogonal ellipse. So far as it lags it possesses a cosine-component, and this contributes to the loop only an oblique line, shearing the loop over ; but this constituent is wattless. Harmonic analysis cannot of itself distinguish as to how much of the fundamental elliptical constituent of the loop, or of the fundamental sine-component of the magnetizing current, is due to eddy-currents, and how much to hysteresis. It is, indeed, already known [*] that the effect of eddy-currents is to widen out the loop elliptically.

§ 10. *Effect of the Higher Sine-Constituents.*

The presence of the third harmonic has been noted in § 7 above. Indeed, it is usually present. A fine example is afforded by the curve, fig. 21, which is taken from Fleming's 'Alternate Current Transformer,' vol. ii. p. 486 (edition of 1892), which affords the following analysis :—

Fig. 21.

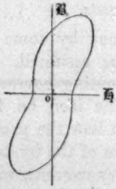


Fig. 22.

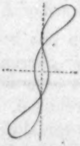
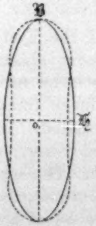


Fig. 23.



$A_1 = 11{\cdot}23$ $B_1 = -11{\cdot}87$
$A_3 = 1{\cdot}71$ $B_3 = -\ 0{\cdot}72$
$A_5 = -\ 0{\cdot}74$ $B_5 = +\ 0{\cdot}23$
$A_7 = +\ 0{\cdot}62$ $B_7 = +\ 0{\cdot}07$
$A_9 = -\ 0{\cdot}29$ $B_9 = +\ 0{\cdot}22$
$A_{11} = +\ 0{\cdot}03$ $B_{11} = +\ 0{\cdot}57$

---

[*] See Ångström, *op. citat.*; and Heinke, *Die Electrophysik* (1904), p. 500.

In the memoir of Ångström already quoted, a curious curve, fig. 22, nearly a pure third harmonic is given, having been found by him as a differential hysteresis loop representing the difference between two steel rods, one having 0·2 per cent. the other 0·8 per cent. of carbon.

It is obvious that the effect of superposing a third-term sine-constituent upon the fundamental ellipse will be to narrow it at the middle. It will widen it toward the ends, the widening beginning at a distance of 0·134 of the semi-major axis from the ends, as in fig. 23.

If sine-constituents were present, of all orders, in the following proportions,

$$A_1 = 1,\ A_3 = \tfrac{1}{3},\ A_5 = \tfrac{1}{5} \ldots\ldots A_n = \frac{1}{n},$$

the form of the loop would become a pure rectangle, as fig. 24.

Fig. 24.

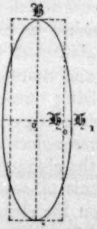
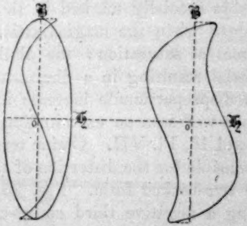


Fig. 25.



The half-width of any static hysteresis loop, which is commonly taken (following Hopkinson) as a measure of the coercive force, is independent of all the cosine terms, and is in all cases equal to $A_1 - A_3 + A_5 - A_7 + A_9 -$ &c.

For loops of equal height (that is equal maxima of flux density) the coercive force is not proportional to the work spent in the cycle; for the work spent in the cycle is represented by the area of the loop, and as shown above, this is invariably proportional to A_1, while the coercive force is represented by the sum of the series named in the previous sentence. Since in static hysteresis loops the fifth and higher terms are practically absent, the coercive force is proportional to $A_1 - A_3$.

§ 11. *Effect of the Cosine-Constituents.*

As already seen, the fundamental cosine-constituent is an oblique line. It is in reality a double line, traversed upwards during half the cycle, and downwards during the other half. The effect of superposing this constituent upon the fundamental ellipse is to shear it over. Cosine-components are always negative, corresponding to lagging constituents of the wave-curve. They shear the $\mathfrak{B}$–$\mathfrak{H}$ curve over toward the right, on the ascending side. If a mass of iron undergoing magnetization is traversed by an air-gap, or is constituted as a non-closed magnetic circuit, the reaction of the air-gap brings a lagging constituent into the magnetizing current, imposing a negative cosine-constituent upon the loop, and shearing it over.

The presence of negative cosine-constituents in hysteresis loops is specially marked in those beaked forms which are obtained when the magnetization has been pushed to high degrees of saturation: the diminished permeability of the material resulting in a diminished reactance, and therefore in a disproportionate increase in the magnetizing current. This is well seen on examination of the analysis of figs. 14, 15, and 16, Pl. VII. Cosine terms of the higher orders are responsible for the distortion of the ellipse into the characteristic two-beaked form. Fig. 25 shows the result of superposing a negative third cosine-constituent upon the ellipse. But this particular figure, resembling a capital **S**, could not result from any experiment, as neither the ascending nor the descending half is single-valued. No experimental curve * could, on ascending from the point where it crosses the $\mathfrak{H}$ axis, curve backwards toward the $\mathfrak{B}$ axis, and then recurve from that axis. Such a curve would be unstable; and the ascending branch could, at most, ascend parallel to the $\mathfrak{B}$ axis before turning to the cusp. Is it too remote to speculate that the vertical portions of the loops experimentally found for soft iron really do represent instabilities, and mask the true forms of the loops?

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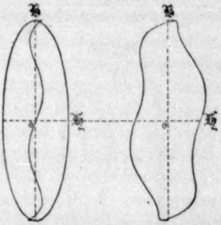
* See, for example, P. Holitscher (*Inaug. Diss.*, Zürich, 1900), "Experimentelle Untersuchungen über den remanenten Magnetismus des Eisens," plate 6.

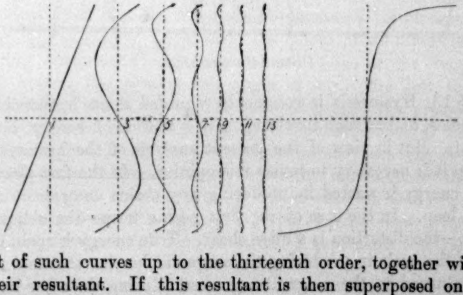
The superposition of a negative fifth-order constituent is shown in fig. 26; but, again, the only possible cases must be such as to yield single-valued resultants.

Fig. 26.



It will have been noticed that in the analyses of some of the loops the amplitudes of the negative cosine terms appear in a descending set of values. If these amplitudes are of relative magnitudes as $1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8}$ &c., the resultant curve will have its middle portion truly vertical. Fig. 27 gives a

Fig. 27.



set of such curves up to the thirteenth order, together with their resultant. If this resultant is then superposed on a fundamental ellipse, the result is a characteristic two-beaked loop.

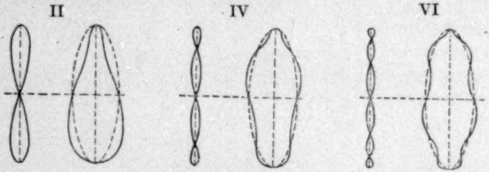
The horizontal length from the cusp to the vertical axis (*i. e.* the amplitude of $\mathfrak{H}$ at its maximum) is in all cases equal to

$$B_1 + B_3 + B_5 + B_7, \text{ \&c.}$$

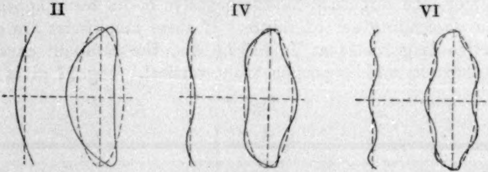
§ 12. *Effect of Harmonic Constituents of Even Orders.*

If constituents of even orders could exist, they would produce distortions of the loop, such as are shown in fig. 28. No such distortions have ever been observed.

Fig. 28.



Sine Components.



Cosine Components.

§ 13. Hysteresis is commonly regarded as an irreversible process, and as such involving a degradation of energy into heat. But in view of the present analysis of the hysteresis loop it is necessary to revise this opinion. In the first place, no energy is wasted in producing any cosine component of the loop. In the case of the first cosine term—the oblique line—the distortion is a mere shear. True energy is spent in half the cycle in producing the magnetic flux, but that energy is returned to the magnetizing circuit during the other half of the cycle, exactly as it is in the case of the production of a magnetic flux in a solenoid devoid of iron. The like is true of the higher cosine-components. Also the sine-components higher than the first represent reversible processes. The only component which represents an irreversible process is the fundamental sine-component, the ellipse, itself due to that component of current which is in phase with the voltage. This irreversible part is due in detail to an energy-waste which at every instant is proportional to the square of the magnetizing current, and is in phase with it. True hysteresis waste accompanies the current, and does not either lag behind it nor precede it. True hysteresis does not cause any lag in the current, being necessarily simultaneous with it. True, the curve of the flux-density always lags 90° behind the voltage curve, and therefore 90° behind the effective component of the current curve. Of the reversible components, it is the first cosine term which is concerned in the lag of the current, and the lagging component of current is wattless. The higher cosine-components conjointly produce additional lags, and are also wattless. The higher sine-components, also wattless, do not cause any lag of the current as a whole. The name "hysteresis" was originally given * to the phenomenon to connote an effect which lagged behind its cause. The term is now usually restricted to the phenomenon of energy-waste. But the energy-waste does not involve or produce any phenomenon of lag. On the contrary, as it is a simultaneous phenomenon, its presence actually produces an advance in the phase of an otherwise lagging current. The greater the reluctance of the magnetic circuit, the greater the angle of lag. The less the permeability which enters into that reluctance the greater is the angle of lag. The lagging components, that is the cosine-components of the loop, so far as they are not due to eddy-currents, depend solely upon the reduction of the permeability of the iron during the process of the cycle of magnetization. The lag is in fact due to components other than the fundamental component which represents the true hysteresis.

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* Ewing, Phil. Trans. 1885, pt. ii. p. 524.

ABSTRACT.

Attempts have been made to find an explanation of the forms of the looped curves which express the hysteresis exhibited by iron and steel when subjected to cycles of magnetization. Physical explanations to account for their general shape have been given by Ewing and Hopkinson, and M. Pierre Weiss has put forward an electronic theory to account for the principal features. The Author shows that any hysteresis loop can be analysed into a harmonic series of closed curves corresponding to the various terms in the analysis of the current wave, and their constituents are examined in the paper. A number of examples of hysteresis loops were chosen and subjected to analysis. The loops chosen related to various kinds of iron and steel, hard and soft, solid and laminated, and taken by various methods. In carrying out the analysis the simple approximate method described by the Author (Proc. Phys. Soc. vol. xiv.) was used. Details are given of the analysis of various loops, the effect of eddy currents on the size and form of the loops is discussed, and an account is given of the effect of the higher sine and cosine constituents of the current wave.

DISCUSSION.

Dr. RUSSELL thanked the Author for his most interesting and instructive paper. In connection with Dr. Thompson's orthogonal ellipse, he showed how an indefinite number of two-beaked loops similar to the hysteresis loop can be made up by quadrants of ellipses, each loop having the same height, the same area, and the same breadth as the orthogonal ellipse. Two-beaked loops having equal areas, breadths, and heights can also be made up from portions of any two curves provided that these curves are similar. He thought it would be of value to discuss the harmonics of the magnetising current, as deduced from the statical hysteresis loop, when the flux varied according to laws other than the harmonic law. F. J. Dykes had investigated the law according to which the flux must vary in order that the magnetising current may have a pure sine shape. To find the harmonics of the flux curve in this case and compare them with the harmonics of the current curve in the Author's case would be of interest. A practical use of Dr. Thompson's rapid method of analysing the current wave into its harmonics is that it enables us to compute approximately the eddy-current losses in iron in certain cases by utilising J. J. Thomson's and Heaviside's formulæ.

Dr. W. H. ECCLES, referring to the emphasis laid by Prof. Thompson on the hysteresis ellipse obtained by his ingenious mode of analysis of the hysteresis loop, suggested that other closed curves would be obtained if other normal functions were used in the analysis. In regard to the Author's suggestion that his results affected our notions of the phenomenon called hysteresis, he thought no change was needed if we kept the term to its original meaning of a lag in state, in contra-distinction to (what is often implied) a lag in time.

XXXI. *The Energy Relations of Certain Detectors used in Wireless Telegraphy.* *By* W. H. ECCLES, *D.Sc., A.R.C.S.*[*]

[Plate VIII.]

THE results of an experimental examination into the physical properties of four very different types of detector used in radio-telegraphy are set forth briefly in the following pages. The conditions of the experiments have been made generally identical with those arising in the ordinary employment of the detectors, and, in particular, the quantities of energy given to the instruments, in the form of electrical oscillations, have been of the same order in these experiments as in actual practice. The detectors investigated are the electrolytic, the carborundum rectifier, the zincite detector, and a thermoelectric detector. All these have before this been subjected to close scrutiny by various observers, who, however, used methods different from that of this paper; a summary of their work will be given alongside the results of the present experiments. These results, it will be seen, are expressed in the form of curves rather than as tables of figures; each curve may be regarded as typical of the detector concerned, and has been selected from a number of curves drawn from measurements accumulated during last year.

The method and apparatus used are the same as were described in a paper "On Coherers," read before the Physical Society in March last (Proc. Phys. Soc. vol. xxii. pt. iii.). The present experiments were for the most part carried out at an earlier date than those described in that paper. The properties of the detectors are examined in three distinct ways. The first way consists in applying to the detector an electromotive force which is gradually increased, and measuring the consequent current at each step. The second way is to fix the electromotive force at some particular value, to send trains of oscillations of various energy values through the instrument, and to measure the intensity of sound produced in the telephone on each occasion. The third way is to send trains of constant energy value through the instrument while the steady electromotive force applied to it is varied in steps, and to measure at each step the intensity of the sound produced in the telephone. These modes of experimenting give curves that may be called respectively the steady-current curve, the power curve, and the sensitiveness curve. It will be seen from the curves that the power supplied to the detector in the form of electrical oscillations and the power handed to the telephone in the form of intermittent current are both recorded in fractions of a watt. For this purpose, the circuit-calibrations described in the former paper were used. It must be mentioned here that the calibration of the telephone circuit is probably much less accurate than that of the detector circuit.

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[*] Read July 8, 1910.

RESULTS OF THE MEASUREMENTS.

The Electrolytic Detector.

This detector consisted of two platinum electrodes in dilute sulphuric acid (one of acid to four of water). One electrode was a platinum wire of 0·0006 cm. diameter drawn by the Wollaston process, dipping a fraction of a millimetre into the electrolyte; the other was a piece of thick wire well immersed. When a potential difference less than one volt is established between the electrodes the current that passes is very small; but as the potential difference is increased the counter electromotive force of polarization is overcome, till finally a large current flows with evolution of gas. The stage of the process which is useful for detecting feeble electrical oscillations is that where the bubbles of gas do not yet form on and break away from the point freely. The steady current curves of fig. 1 (Pl. VIII.) indicate the difference between the two cases, point as anode and point as cathode. The power curves for various cases are collected in fig. 2, all from the same detector. Curve *a* shows the relation between the power *w* delivered by the detector to the telephone and the power W given in the form of electrical oscillations to the detector, when the potential difference between the platinum point and the large electrode has the value 2·9 volts—the voltage of highest sensitiveness. Curve *b* shows the large fall in sensitiveness caused by altering the applied electromotive force, the point still being anode.

In the same way, curves *c* show the efficiency of the energy transformation when the point is negative. Curve *c* is an attempt to reach the best possible sensitiveness with the point as cathode; but it is to be remarked that when the point is covered with hydrogen, the electrical conditions are somewhat unstable, and the best potential difference is an uncertain quantity. This is in strong contrast with the very definite conditions that rule when the platinum point is polarized with oxygen. This is clearly indicated by the curves of fig. 3, where the ordinates represent the proportion of energy delivered to the telephone when the applied voltage has various values.

This detector has received a very great deal of attention in the past, but the precise mode of operation of the instrument is still unknown. Reich[*], after making experiments with superposed direct current and alternating current of low frequency, considered that the phenomena could only be accounted for by a dissolution of the small electrode used as anode and a simultaneous disappearance of oxygen—processes purely chemical. Rothmund and Lessing[†], using a Blondlot oscillator and Lecher wires, and measuring the potential difference across the detector and the current through it when the oscillator was working and not working, concluded that the whole action of the instrument depended upon some unexplained "depolarization action" of the oscillations; and by using a variety of electrolytes, proved the incorrectness of the hypothesis, which was a mere surmise unsupported by scientific measurements, that the instrument operated by resistance alterations due to the heat generated in the liquid mass near the minute anode. Later Dieckmann[‡] measured the current changes that followed upon the passage of strong oscillations of (unmeasured) intensity through a detector. Later still, Austin[§], using alternating current of low frequency, showed that the detector was affected by electromotive amplitudes of 1/10000 volt, and that oscillations produced by spark-discharges in the laboratory affected the detector equally whether the small electrode was anode or cathode. He concluded that in the action of the instrument heat had a share, and that chemical action, electrostatic attraction across the gas film, and also a property styled rectification, all took part.

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[*] *Phys. Zeitschr.* v. p. 338 (1904).

[†] *Ann. d. Phys.* xv. 1, p. 193 (1904).

[‡] *Phys. Zeitschr.* v. p. 529 (1904).

[§] Bulletin, Bureau of Standards, i. 3, p. 435 (1905).

Carborundum.—The detector was set up by clamping a crystal of carborundum between brass plates, so that a smooth crystalline edge or corner was in contact with one plate, and a blunt and more amorphous part of the crystal in contact with the other plate. The steady current curves of two crystals widely different in their electrical behaviour appear in fig. 4 (Pl. VIII.). Curves *a b* belong to one crystal, the dotted curves AB belong to another. The upper curve of each pair was obtained when the jagged blunt end of the crystal was positive.

The power curves are given in fig. 5. Lines *a b c* exhibit the energy relations for the crystal that gave *a b* in fig. 4. Line *a* was obtained while the blunt end was at a potential 2·62 volt higher than the smooth end ; *b* was got when the potential difference was −0·44 ; curve *c* was got without electromotive force. Lines A, B, C refer to the other crystal. Line A was obtained while the blunt end was 2·9 volt above the smooth end ; line B while the blunt end was 2·1 volt below ; line C while no external electromotive force was applied. The curve of fig. 6 shows how sensitiveness altered with the electromotive force applied to the terminals of the detector.

The Carborundum detector has been examined very exhaustively by Pierce[*]. He has shown that crystals of this substance may be as much as 1000 times more conductive for current in one direction than in the opposite, and has concluded that the substance acts as a detector of high frequency oscillations solely because of this unilateral conductivity. Heat, he considered, played no part in the process. The curves given above show, however, that a crystal may be a good detector even though its unilateral conductivity be not very pronounced.

---

[*] Phys. Review, xxv. p. 31 (1907).

Zincite-chalcopyrite.—The detector made by arranging a corner of a fragment of brown zincite (native oxide of zinc) to press against a piece of chalcopyrite (iron copper sulphide) is one of the most sensitive known. It is used extensively in various navies. Fig. 7 gives the results of measurements of current under steady electromotive force. For this particular detector the most sensitive condition was attained when the zincite was maintained at a potential about 0·45 volt below that of the pyrite. The power curves are given in fig. 8; here *a* is the curve when no electromotive force was applied: *b* is the curve for an applied electromotive force of 0·45 volt, zincite negative, and *c* is the curve for an applied electromotive force of 0·45 volt, zincite positive. It will be seen from these that the combination forms a very sensitive detector, even when no external electromotive force is applied. The curves showing the change in sensitiveness with variation of the applied electromotive force are plotted in fig. 9.

Graphite-Galena.—A detector that is very widely used for every-day telegraphy is that consisting of a pointed piece of graphite touching the face of a crystal of galena. The curves connecting applied electromotive force and current flowing through the contact appear in fig. 10. The power curves are shown in fig. 11: curve *a* is obtained when the external electromotive force is not applied, and curves *b* and *c* when electromotive forces of 0·45 volt and −0·45 volt were applied. The connexion between the power given to the telephone and the electromotive force applied to the detector is given in fig. 12.

These two last detectors and others similar to them are sometimes called "rectifiers," sometimes "thermoelectric detectors." They are styled thermoelectric because it was originally supposed that they owed their power of detecting high frequency vibrations to the thermoelectromotive forces set up at the contact by the rise of temperature produced at that point—the point of highest resistance in the whole oscillation circuit—in obedience to Joule's law; but most observers have concluded from experiments with both direct and alternating currents, that these detectors derive their function from an unexplained and hitherto unknown power of rectifying rather than from a combination of the Joule and Peltier effects. Pierce[*] has examined the behaviour of contacts made with anastase, brookite, and molybdenite under alternating currents of ordinary frequency, and obtained oscillograms of the current through them. No evidence of thermoelectric or other integrative action was perceived in the photographs. Austin[†] has examined quantitatively, also by aid of slow alternating currents, the properties of detectors consisting of contacts of silicon and steel, carbon and steel, tellurium and aluminium. Brandes[‡] and Raetenkrantz[§] have also contributed greatly to our knowledge of the connexion between the steady current curve of a detector and its behaviour under electrical oscillations.

Conclusion.

The chief fact brought to light by the above experiments is that the energy passed to the telephone by a detector is connected linearly with the energy given to the detector in the form of electrical oscillations. This is true for all the detectors examined, even including the coherers discussed in the earlier paper. The curves connecting the input and output of energy though they are straight lines usually pass some distance away from the origin. This implies that for a particular detector under invariable conditions there is a fixed wastage of oscillation energy, amounting commonly to about 1/10 of an erg per second, however large or small the oscillation energy given to the detector may be. Another interpretation is, however, that a small quantity of energy, which is invariable while the detector is undisturbed, is delivered by the detector to the telephone circuit in a form that never makes any proportion of itself manifest as sound. The curves suggest, though they do not prove, that all detectors are fundamentally thermal in their action. That this deduction is opposed to the conclusions reached by previous experimenters is clear from the summary of their work given above. The principal cause of this difference between our conclusions appears to be that nearly all previous observers have used comparatively large quantities of oscillation energy, and have therefore probably brought into play phenomena that never arise in detectors as used in wireless telegraphy.

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[*] Phys. Review, p. 153 (1909).

[†] Bulletin, Bureau of Standards, v. p. 133 (1908).

[‡] *Elektro. Zeitschr.* xxvii. p. 1015 (1906).

[§] *Phys. Zeitschr.* ix. p. 911 (1908).

The above investigations were carried out by the aid of a grant from the Royal Society's Government Grant Committee.

ABSTRACT.

The paper is a record of the results of an experimental examination into the physical properties of the electrolytic detector, the zincite rectifier, the carborundum rectifier, and a thermoelectric detector consisting of a light contact between graphite and galena. The conditions of the experiments have been generally identical with those arising in the ordinary employment of the detectors, and, in particular, the quantities of energy given to the instruments, in the form of electrical oscillations, have been of the same order in these experiments as in actual practice. Three ways of investigation are followed. The first way consists in applying to the detector an electromotive force which is gradually increased, and measuring the current at each step. The second way is to fix the electromotive force at some particular value, to send trains of oscillations of various energy values through the instrument, and to measure the intensity of the sound produced in the telephone on each occasion. The third way is to send trains of constant energy value through the instrument while the steady electromotive applied to it is varied, and to measnre at each step the intensity of the sound produced in the telephone. These modes of investigation give curves that may be called respectively the steady current curve, the power curve, and the sensitiveness curve. The chief fact brought to light is that the power curves of all the detectors are straight lines, which suggests that all the detectors are fundamentally thermal in their action.

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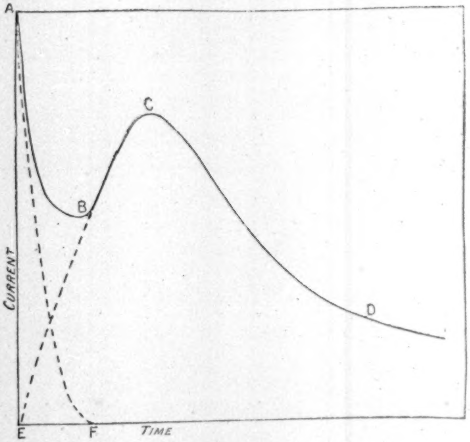
FIG. 2.



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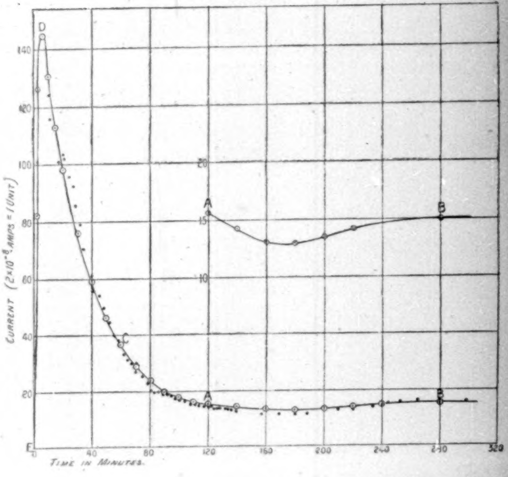

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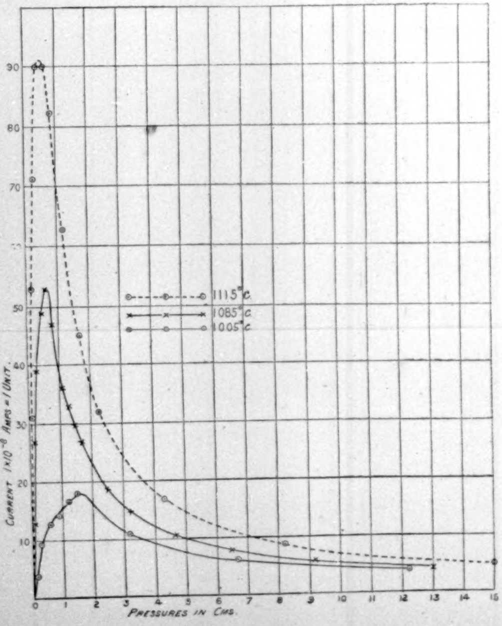

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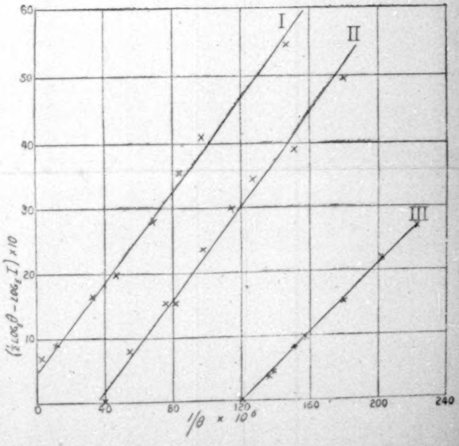

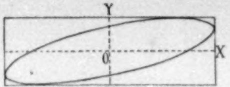

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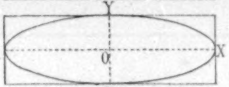

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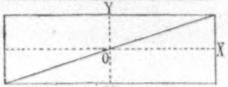

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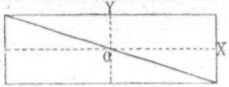

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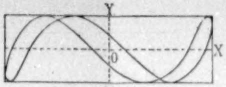

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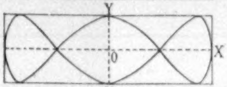

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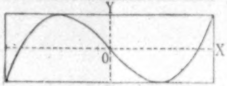

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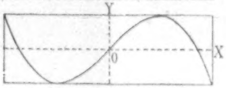

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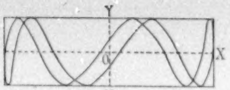

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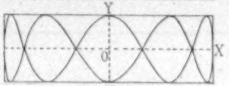

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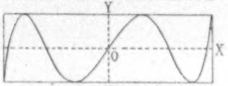

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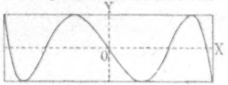

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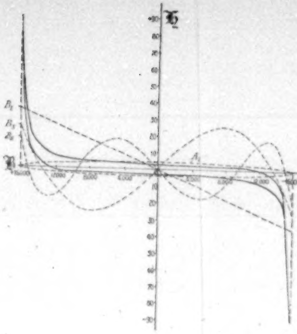

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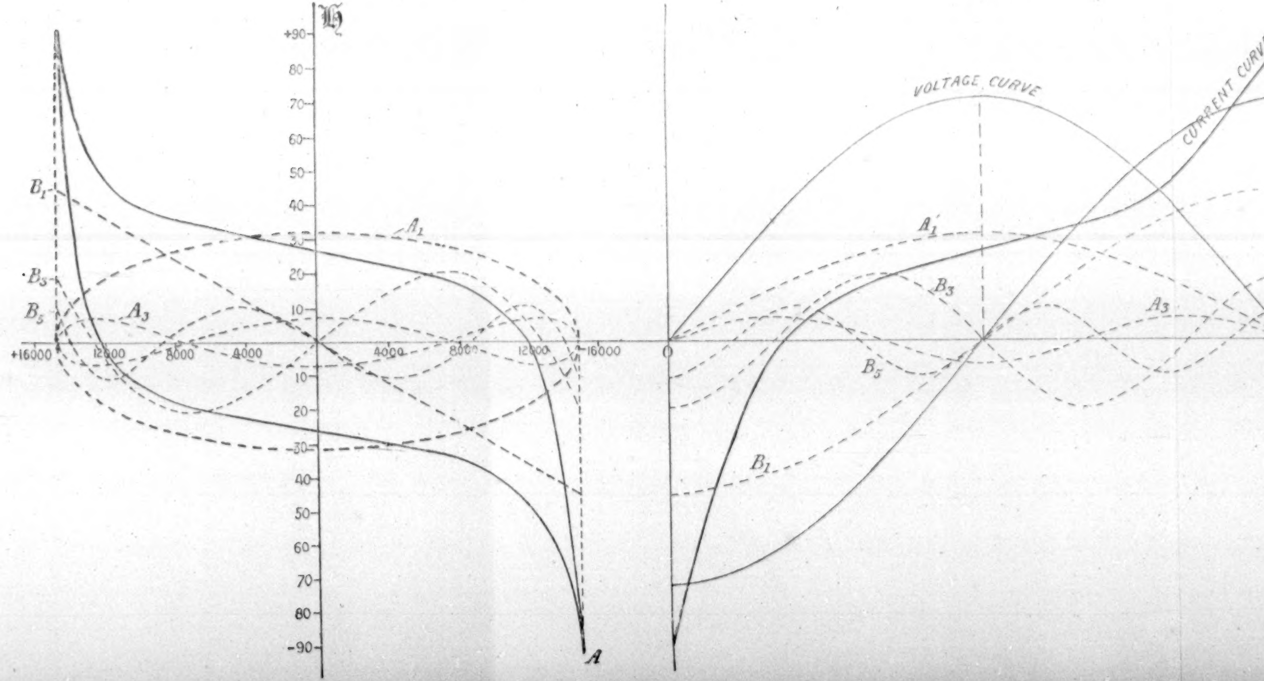

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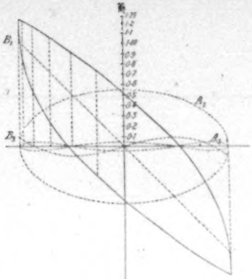

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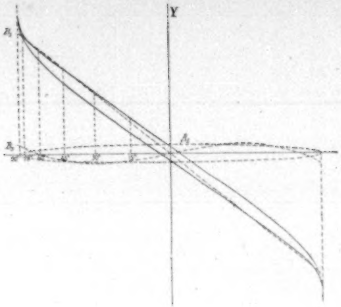

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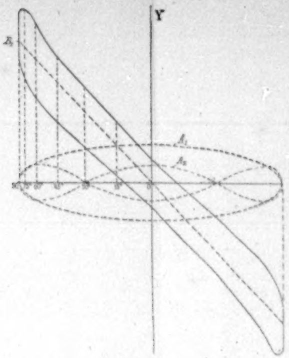

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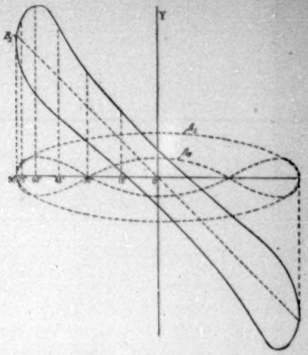

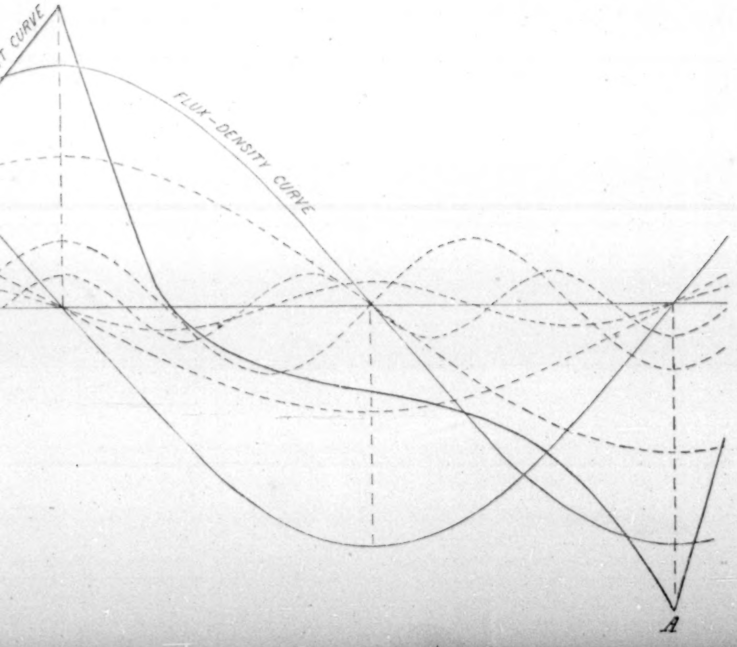

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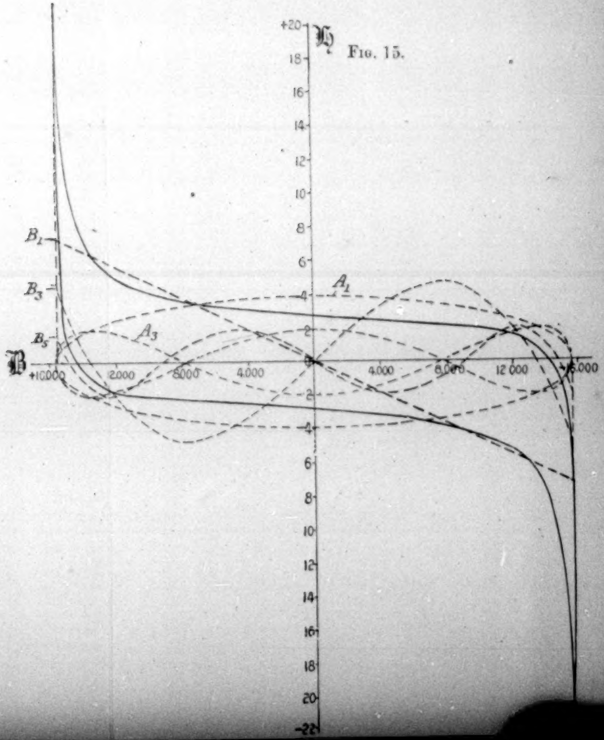


ELECTROLYTIC DETECTOR.

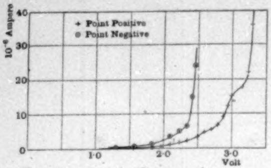


Fig. 1.—Steady Current Curves.

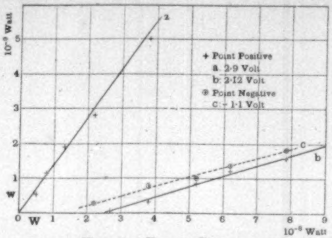


Fig. 2.—Power Curves.

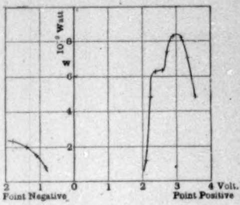


Fig. 3.—Sensitiveness Curve.
$W = 6{\cdot}2 \times 10^{-7}$ Watt.

CARBORUNDUM DETECTOR.

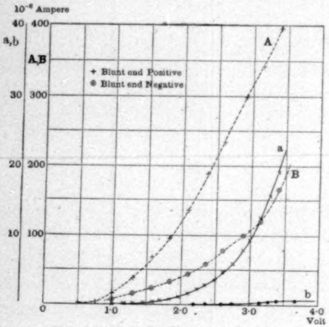
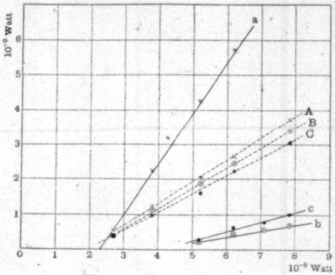


Fig. 4.—Steady Current Curves.



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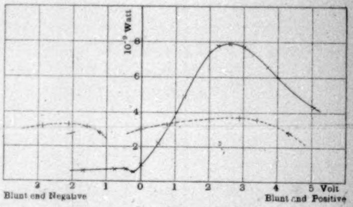

Fig. 6.—Sensitiveness Curves.
$W = 7{\cdot}85 \times 10^{-8}$ Watt.

ZINCITE-CHALCOPYRITE DETECTOR.

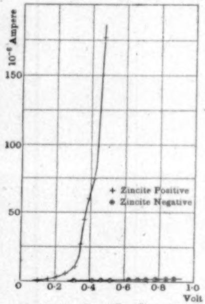

Fig. 7.—Steady Current Curves.

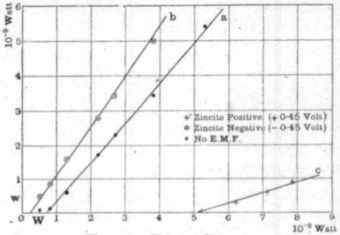

Fig. 8.—Power Curves.

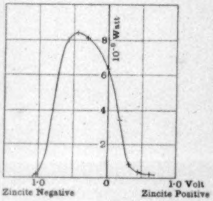

Fig. 9.—Sensitiveness Curve.
$W = 6{\cdot}2 \times 10^{-8}$ Watt.

GRAPHITE-GALENA DETECTOR.

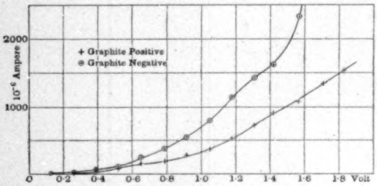

Fig. 10.—Steady Current Curves.

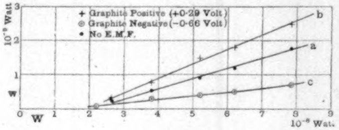

Fig. 11.—Power Curves.

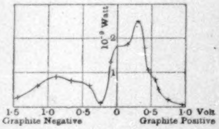

Fig. 12.—Sensitiveness Curve
$W = 7{\cdot}85 \times 10^{-8}$ Watt.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1909-1910.

February 26th, 1909.

Meeting held at the Finsbury Technical College by kind invitation of Prof. S. P. Thompson, F.R.S., and Prof. E. G. Coker.

Dr. C. Chree, F.R.S., President, in the Chair.

The new Engineering Laboratory was open to Fellows for inspection.

The following Papers were read:—

1. A Laboratory Machine for applying Bending and Twisting Moments simultaneously. By Prof. E. G. Coker.

2. On the Self-demagnetizing Factor of Bar Magnets. By Prof. S. P. Thompson, F.R.S., and E. W. Moss.

An exhibition of the optical properties of combinations of mica and selenite films (after Reusch and others) in convergent polarized light was given by Prof. S. P. Thompson, F.R.S.

Mr. C. R. Darling exhibited the following:—

An experiment to illustrate the temperature of equal density of aniline and water.

A simple form of thermo-electric pyrometer for students' use.

A combined metre-bridge and potentiometer, with new tapping-key device, for pyrometric and general laboratory work.

A new form of carbon-plate rheostat, suitable for control of small electric furnaces.

March 12th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Candidates were elected Fellows of the Society:—
Prof. P. V. Bevan, J. W. Bispham, L. Blaikie, Prof. E. G. Coker, A. Eagle, Sir R. A. Hadfield, F. J. Harlow, C. F. Hogley, B. M. Narbeth, T. Smith, and Prof. The Hon. R. J. Strutt.

The following Papers were read:—

1. The Effect of Radiations on the Brush Discharge. By A. E. Garrett.
2. Pirani's Method of Measuring the Self-Inductance of a Coil. By E. C. Snow.
3. Exhibition of a High-Potential Primary Battery. By W. S. Tucker.
4. On the Least Moment of Inertia of an Angle-bar Section. By H. S. Rowell.

March 26th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

Mr. F. Lloyd Hopwood was elected a Fellow of the Society.

The following Papers were read:—

1. The Production of Steady Electrical Oscillations in Closed Circuits, and a Method of Testing Radiotelegraphic Receivers. By Dr. J. A. Fleming, F.R.S., and G. B. Dyke.
2. The Effect of an Air Blast upon the Spark Discharge of a Condenser charged by an Induction Coil or Transformer. By Dr. J. A. Fleming, F.R.S., and H. W. Richardson.
3. On the Action between Metals and Acids and the Conditions under which Mercury causes Evolution of Hydrogen. By Dr. S. W. J. Smith.

April 23rd, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read:—

1. On a want of Symmetry shown by Secondary X-Rays. By Prof. W. H. BRAGG, F.R.S., and J. L. GLASSON.

2. Transformers of X-Rays. By C. A. SADLER.

3. Theory of the Alternate Current Generator. By Prof. T. R. LYLE.

Prof. L. R. WILBERFORCE exhibited a galvanometer of the Broca pattern suitable for general use in elementary laboratories.

May 14th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read:—

1. A Bifilar Vibration Galvanometer. By W. DUDDELL, F.R.S.

2. Effect of Temperature on the Hysteresis Loss in Iron in a Rotating Field. By W. P. FULLER and H. GRACE.

3. On a Method of Testing Photographic Shutters. By A. CAMPBELL and T. SMITH.

June 11th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read:—

1. The Arthur Wright Electrical Device for evaluating Formulæ and solving Equations. By Dr. A. RUSSELL and ARTHUR WRIGHT.

2. The proposed International Unit of Candle Power. By C. C. PATERSON.

3. Inductance and Resistance in Telephone and other Circuits. By Dr. J. W. NICHOLSON.

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A "Note on Terrestrial Magnetism" by Mr. G. W. WALKER, and a paper by Mr. A. EAGLE, "On the Form of the Pulses constituting full Radiation or White Light," were taken as read.

June 25th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read :—

1. A Transition Point in Zinc Amalgam. By Prof. H. S. CARHART.

2. A Method of producing an intense Cadmium Spectrum, with a proposal for the use of Mercury and Cadmium as Standards in Refractometry. By T. M. LOWRY.

3. On the Measurement of Wave Length for High Frequency Electrical Oscillations. By A. CAMPBELL.

4. An Electro-Magnetic Method of Studying the Theory of and Solving Algebraical Equations of any Degree. By Dr. RUSSELL and J. N. ALTY.

5. The Sine Condition in relation to the Coma of Optical Systems. By S. D. CHALMERS.

6. An Instrument for Measuring the Strength of an Intense Horizontal Mognetic Field. By F. W. JORDAN.

Dr. C. V. DRYSDALE exhibited a new Féry Thermo-electric Calorimeter.

The following Papers were taken as read :—

7. On a Method of Determining the Sensibility of a Balance. By Prof. POYNTING and G. W. TODD.

8. The Balance as a Sensitive Barometer. By G. W. TODD.

October 22nd, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. CHREE, F.R.S., President, in the Chair.

The following Papers were read :—

1. On Cadmium Amalgams and the Weston Normal Cell. By F. E. SMITH.

2. Production of Radium from Uranium. By F. Soddy.
3. The Rays and Product of Uranium X. By F. Soddy.
4. The Production of Helium from Uranium and Thorium. By F. Soddy.

November 12th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Papers were read:—

1. The Absorption Spectrum of Potassium Vapour. By Prof. P. V. Bevan.
2. Some further Notes on the Physiological Principles underlying the Flicker Photometer. By J. S. Dow.
3. Tables of the ber and bei and ker and kei Functions, with further formulæ for their computation. By H. G. Savidge: was taken as read.

Dr. Edridge Green exhibited a "Colour-Perception Spectrometer."

November 26th, 1909.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Papers were read:—

1. The Effective Resistance and Inductance of a Helical Coil. By Dr. J. W. Nicholson.
2. Ductile Materials under Combined Stress. By W. A. Scoble.
3. The Recoil of Radium C from Radium B. By Drs. W. Makower and S. Russ.

A paper on "The Sun's Motion with respect to the Æther," by Dr. C. V. Burton, was taken as read.

December 14th, 1909.

An Exhibition of Apparatus was held at the Imperial College of Science and Technology, the following firms taking part:—

Messrs. Bausch & Lomb Optical Co.; British Radio-Telegraph and Telephone Co. Ltd.; Cambridge Scientific Instrument Co.;

Casella & Co.; A. C. Cossor Ltd.; G. Cussons Ltd.; J. H. Dallmeyer Ltd.; Elliott Bros.; Everett, Edgcumbe & Co.; Gallenkamp & Co.; Gambrell Bros.; F. Harrison Glew; J. J. Griffin & Sons; Hicks & Co.; A. Hilger Ltd.; India Rubber, Gutta Percha & Telegraph Works Co., Ltd.; Isenthal & Co.; Marconi's Wireless Telegraph Co.; Leslie Miller; Muirhead & Co.; Nalder Bros. & Co.; Negretti & Zambra; Newton & Co.; R. W. Paul; The Reason Manufacturing Co.; Ross Ltd.; Siemens Bros. & Co., Ltd.; Snell & Tinsley; Synchronome Co.; Weston Electrical Instrument Co.; Carl Zeiss Ltd.

The following short discourses were given:—

1. Experiments on Soap Bubbles. By Prof. C. V. Boys, F.R.S.
2. Some Combinations of Mica and Selenite Crystals. By Prof. Silvanus P. Thompson, F.R.S.

January 21st, 1910.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Papers were read:—

1. Saturation Specific Heats &c., with van der Waals' and Clausius' Characteristics. By R. E. Baynes.
2. The Polarization of Dielectrics in a Steady Field of Force. By Prof. W. M. Thornton.
3. On the Use of Mutual Inductometers. By A. Campbell.

Annual General Meeting.

February 11th, 1910.

Meeting held at the Imperial College of Science and Technology.

Dr. C. Chree, F.R.S., President, in the Chair.

The following Report of the Council was read by the Secretary:—

Since the last Annual General Meeting eleven ordinary Science Meetings and one informal Meeting of the Society have been held at the Imperial College of Science and Technology. The average attendance at the Meetings, apart from the Annual Exhibition, has been 41 as compared with 37 last session.

The Fifth Annual Exhibition of Apparatus by Manufacturers was held on December 14th. A departure from the usual programme was made on this occasion. The Exhibition was held on a Tuesday instead of a Friday, so that it might be open on both the afternoon and evening, and experimental demonstrations were given by Prof. C. V. Boys, F.R.S., and Prof. S. P. Thompson, F.R.S. There were thirty-seven exhibitors. The new arrangements seem to have been appreciated, as there was a very much larger attendance than hitherto, the number of Fellows and Visitors (including exhibitors) amounting to about 700.

The number of Ordinary Fellows now on the roll, as distinct from Honorary Fellows, is 436, an increase of 6 on the number last year; 21 new Fellows have been elected. Two Honorary Fellows were elected at the last Annual General Meeting, namely Mons. R. Benoit and Prof. Julius Thomsen. There have been four resignations, and five Fellows have been struck off the register for non-payment of subscriptions. The Society has to mourn the loss by death of two Honorary Fellows, namely, Prof. F. Kohlrausch and Prof. Simon Newcomb; one Past President, Dr. Shelford Bidwell, and four other Ordinary Fellows, namely, H. E. Harrison, Ludwig Mond, F.R.S., W. J. Russell, F.R.S., and W. F. Stanley, to the last of whom the Society is indebted for funds enabling them to issue the Bulletin.

The Report of the Council was adopted.

The Report of the Treasurer and the Balance Sheet were presented and adopted.

The following were elected Honorary Fellows of the Society:—
Prof. S. ARRHENIUS, Madame CURIE, and Prof. G. E. HALE.

The election of Officers and other Members of Council then took place, the new Council being constituted as follows:—

President.—Prof. H. L. CALLENDAR, M.A., F.R.S.

Vice-Presidents who have filled the Office of President.—Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; Prof. A. W. REINOLD, M.A., F.R.S.; Prof. Sir ARTHUR W. RÜCKER, M.A., D.Sc., F.R.S.; Sir W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.; Prin. Sir OLIVER J. LODGE, D.Sc., LL.D., F.R.S.; Prof. SILVANUS P. THOMPSON, D.Sc., F.R.S.; R. T. GLAZEBROOK, D.Sc., F.R.S.; Prof. J. H. POYNTING, Sc.D., F.R.S.; Prof. J. PERRY, D.Sc., F.R.S.; C. CHREE, Sc.D., LL.D., F.R.S.

Vice-Presidents.—A. CAMPBELL, B.A.; Prof. C. H. LEES, D.Sc., F.R.S.; Prof. A. SCHUSTER, Ph.D., F.R.S.; S. SKINNER, M.A.

Secretaries.—W. R. COOPER, M.A.; S. W. J. SMITH, M.A., D.Sc.

Foreign Secretary.—Prof. S. P. THOMPSON, D.Sc., F.R.S.

Treasurer.—W. DUDDELL, F.R.S.

Librarian.—W. WATSON, D.Sc., F.R.S.

Other Members of Council.—Prof. W. H. BRAGG, M.A., F.R.S.; W. H. ECCLES, D.Sc.; A. GRIFFITHS, D.Sc.; J. A. HARKER, D.Sc., F.R.S.; T. MATHER, F.R.S.; A. RUSSELL, M.A., D.Sc.; W. N. SHAW, M.A., D.Sc., F.R.S.; F. E. SMITH; R. S. WHIPPLE; and R. S. WILLOWS, M.A., D.Sc.

Votes of thanks were passed to the Auditors, to the Officers and Council, and to the Governors of the Imperial College of Science and Technology.

Prof. H. L. CALLENDAR then took the Chair and delivered an Address.

TREASURER'S REPORT.

POSSIBLY owing to the amendment of Article 26 of the Memorandum of Association, the year 1909 has been a record year for the receipt of subscriptions, the amount collected exceeding that realised in 1908 by over £150. At the same time the printer's bill has swollen in a similar proportion, owing to the Secretary's efforts to expedite publication, so that the net balance on current account has increased by only £22. The assets of the Society show no material change, but a larger margin of income over expenditure is desirable.

HUGH L. CALLENDAR,
Hon. Treasurer.

The Treasurer in Account with the Physical Society, from January 1st, 1909, to December 31st, 1909.

Dr.	£	s.	d.	£	s.	d.
Jan. 1, 1909. Balance in Bank				56	6	6
Entrance-Fees	21	0	0			
Subscriptions	486	2	2			
Extra Subscriptions	27	3	0			
Arrears	35	14	0			
Abstracts etc.	12	5	0			
Students	6	7	4			
				588	11	6
Composition-Fees				53	11	0
Dividends:—						
Furness Debenture Stock	15	3	2			
Metropolitan Board of Works	6	12	6			
Lancaster Corporation Stock	11	7	0			
New South Wales	8	8	3			
Midland Railway	37	18	4			
Great Eastern Railway	18	19	2			
London, Brighton, & South Coast Railway	23	14	6			
				122	2	11
Sales of Publications				42	19	0
Interest on £400 deposit account				6	8	0
				£869	18	11

Cr.	£	s.	d.	£	s.	d.
Science Abstracts				279	19	6
Taylor and Francis:—						
Proceedings	218	14	0			
Abstracts, etc.	147	4	11			
Distribution	27	2	6			
Separate Copies	30	10	6			
				423	11	11
Periodicals				3	17	0
Binding				1	10	0
Reporting				35	0	0
Attendance				16	5	0
Refreshments				9	18	6
Petty Cash (Secretaries)				14	0	0
Commissions					2	5
Subscriptions paid in error				5	3	0
Royal Astronomical Society				2	2	0
Balance in Bank				78	9	7
				£869	18	11

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

12th January, 1910.

A. A. C. SWINTON, } *Auditors.*
A. WRIGHT,

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31, 1909.

ASSETS.	£	s.	d.
Subscriptions due, Treasurer's estimate	20	0	0
£533 Furness Ry. Co. 3 per cent. Debenture Stock	416	0	0
£1600 Midland Railway 2½ per cent. Preference Stock	1092	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	201	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	332	0	0
£254 2*s.* 9*d.* New South Wales 3½ per cent. Inscribed Stock	249	0	0
£500 London, Brighton, and South Coast Railway Ordinary Stock	530	0	0
£500 Great Eastern Railway 4 per cent. Debenture Stock	535	0	0
Balance in Bank	78	9	7
Ditto on deposit	400	0	0
Publications in Stock, estimated	200	0	0
	£4053	9	7

LIABILITIES.	£	s.	d.
Cheque not presented	1	3	6
Balance	4052	6	1
	£4053	9	7

Audited and found correct,

HUGH L. CALLENDAR, *Honorary Treasurer.*

12*th January*, 1910.

A. A. C. SWINTON, } *Auditors.*
A. WRIGHT,

PRESIDENT'S ADDRESS.

Before dealing with the special subject of my Address, it is my sad duty to recount briefly the losses sustained by the Society since the last General Meeting. The losses include two of the most distinguished of our Foreign Fellows, and five on the home list—one of whom was a past President of the Society.

By the death of Simon Newcomb the United States have lost their most famous astronomer and mathematician. In Friedrich Kohlrausch the world has lost a physicist, whose name was a household word in every laboratory. These men belong to the world of science more than to our own Society, and cannot be adequately commemorated here.

Dr. Shelford Bidwell, F.R.S., joined the Physical Society in 1877, soon after its foundation, and became President in 1897. In his presidential address of 1898 he attributed his pursuit of science largely to the interest which he found and the friends he made attending meetings of the Physical Society. As a physicist he found a singular fascination in unravelling obscure and apparently paradoxical phenomena. His investigation on the photoelectric properties of selenium, and on the action of microphonic contacts are striking examples of attention to minute but important detail. One of his best known researches was that on the magnetic expansion of iron, and on magnetic traction, which he enriched with many original observations and methods. In later years he devoted himself to the subjective phenomena of vision, especially ghosts and after-images, which he described in a book called "Curiosities of Light and Sight," published in 1899 at the conclusion of his term of office as president. It is characteristic of the man that his devotion to this research very nearly cost him his eyesight. The Physical Society may well be proud to claim a share in kindling the enthusiasm which led to his achievements.

William Ford Stanley joined the Society in 1882, and at one time served on the Council. He was most successful and distinguished as a maker and designer of scientific instruments, and turned his success to good account in many philanthropic enterprises. The Physical Society is particularly indebted to him for the munificent donation of £500 stock, the largest ever received by the Society, to aid in the publication of the 'Bulletin,' which has proved in recent years most useful in increasing the interest of our meetings.

HUGH ERAT HARRISON, who joined the Society in 1882, was better known as an electrical engineer than a physicist He was for twenty years Principal of Faraday House, the great electrical training and testing institution, in the successful organization of which he had a very large share, and in which he is succeeded by another member of our Society.

Dr. LUDWIG MOND, F.R.S., was one of the leading chemical technologists of the world, and has earned the gratitude of all scientific workers by his munificent donations for the advancement of pure science. Although he did not take an active part in the special work of our Society, physicists have indirectly benefited in many ways by his liberality.

WILLIAM JAMES RUSSELL, F.R.S., who joined in 1878, was one of the oldest members of the Society. He was previously a chemist, but much of his recent work on obscure photographic effects was of a character which appealed also to the physicist.

Here follows special subject of Address which will be found
on pp. 220–252 of the 'Proceedings.'

February 25th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Paper was read:—

1. Telephone Circuits. By Prof. J. Perry, F.R.S.

The following Papers were taken as read:—

2. On the Laws regarding the direction of Thermoelectric Currents enunciated by M. Thomas. By Prof. C. H. Lees, F.R.S.

3. A new Method of determining Thermal Conductivity. By H. R. Nettleton.

March 11th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read:—

1. On Coherers. By Dr. W. H. Eccles.

2. Earth-air Electric Currents. By Dr. G. C. Simpson.

3. An automatic Toepler Pump designed to collect the gas from the apparatus being exhausted. By Dr. B. D. Steele.

April 8th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Paper was read:—

An experimental demonstration of the Loading of Artificial Telephone Cables. By B. S. Cohen.

April 22nd, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read:—

1. Further Tests of Brittle Materials under Combined Stress. By W. A. Scoble, B.Sc.

2. The Magnetic Balance of Curie and Cheneveau. By C. CHENEVEAU, with an Appendix by A. C. JOLLEY.

May 27th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. CALLENDAR, F.R.S., President, in the Chair.

The following Papers were read:—

1. On an Oscillation Detector actuated solely by Resistance-Temperature Variations. By Dr. W. H. ECCLES.

2. The Limitations of the Weston Cell as a Standard of Electromotive Force. By Dr. S. W. J. SMITH.

Mr. A. EAGLE, B.Sc., exhibited a Resonance Transformer.

June 10th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. CALLENDAR, F.R.S., President, in the Chair.

The following Papers were read:—

1. A Galvanometer for Alternate Current Circuits. By Dr. W. E. SUMPNER and W. C. S. PHILLIPS, B.Sc.

2. The Positive Electrification due to Heating Aluminium Phosphate. By A. E. GARRETT, B.Sc.

June 24th, 1910.

An informal Meeting was held at the National Physical Laboratory, where a number of demonstrations were given and Fellows had an opportunity of inspecting the laboratories.

July 8th, 1910.

Meeting held at the Imperial College of Science and Technology.

Prof. H. L. Callendar, F.R.S., President, in the Chair.

The following Papers were read:—

1. The Radio-balance: a Thermoelectric Balance for the Absolute Measurement of Radiation; with applications to Radium and its Emanation. By Prof. H. L. Callendar, F.R.S., President.

2. The Convection of Heat from a Body cooled by a Stream of Fluid. By Dr. A. Russell.

3. Hysteresis Loops and Lissajous' Figures, and on the Energy wasted in a Hysteresis Loop. By Prof. S. P. Thompson, F.R.S.

The following Paper was taken as read:—

4. The Energy Relations of Certain Detectors used in Wireless Telegraphy. By Dr. W. H. Eccles.